Monitoring First and Second Order Statistical Data for Gauging Voltage Collapse Proximity

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***Abstract*—Increasingly, large scale power networks are being operated closer and closer to system stability limits in order to maximize constrained infrastructure. As these systems approach the brink of maximum power transfer, voltage collapse becomes an increasingly realistic threat. Prior research has shown that as a power system approaches a saddle node bifurcation, the variance and autocorrelation statistics of current and voltage time series data begin to rise dramatically [2]. With the increasing deployment of PMUs around the grid, time series data is available to grid operators in real time. Therefore, these statistics can potentially provide an important glimpse into the dynamic and voltage stability of a power system. They might also serve as effective early warning signs for a looming bifurcation. In this paper, we first present a useful framework for quantifying the strength of the load noise associated with a load bus. Next, we then present a method for calculating a critical bus voltage variance which should not be exceeded. If this critical variance is exceeded, grid operators can be warned to take mitigating control action.**

***Index Terms*—Power System Stability, Voltage Collapse, PMU**

I. Voltage Collapse: Brief Overview and Introduction

On July 23 of 1987, voltage collapse caused a large scale blackout to occur in the Tokyo, Japan. Over 2.8 million households serviced by the Tokyo Electric Power Company, Inc. (TEPCO) lost electrical power for a period of over 3 hours. Ultimately, this blackout was caused by severe voltage collapse, and the event has shaped TEPCO's current operational planning and reactive power control methods [4]. Voltage collapse is a phenomena which occurs in heavily loaded power systems, and it is manifested through sagging bus voltages which begin to accelerate downwards very rapidly. Voltage collapse can be attributed to a variety of factors [1], including heavy system loading, limited transmission capacity, generator reactive power production, reactive power shortages, tap changing transformer limits, and generator dynamics (equilibrium points). Voltage collapse falls into the "long term voltage stability" category of system stability, and it is studied and responded to on the time scale of several minutes [3].

II. Using PMU Data to Quantifying Load Noise

Loads throughout a power system are inherently stochastic: operators are unable to precisely predict how the loads are going to change from moment to moment. Over a sufficiently short interval, though, it is safe to assume that the load mean is relatively constant. Mathematically, the real power can be modeled as a function of t, where $P(t) \approx P_0(1+\eta(t))$, where P_0 is the base load, and $\eta(t)$ is a stochastic noise injection with the mean $\mathrm{E}[\eta(t)] = 0$ and variance $\mathrm{Var}(\eta(t)) = \sigma_\eta^2$. Via state estimation, grid operators are able to quantify the size (mean) of the loads around the network every few minutes. The variance of the loads, though, is something which the operators typically do not compute or consider for stability analysis. In the following section, we present a method for quantifying load noise.

A. Functions of Random Variables

Often, a random variable, W, can be expressed as the function of another random variable, X. This relationship is given by (1).

$$W = \phi(X) \tag{1}$$

Given that the PDF of X is $f_X(x)$, and given that ϕ is a monotonically changing (increasing or decreasing) function, the distribution of W can be computed:

$$f_W(w) = \left| \frac{d\psi(w)}{dw} \right| f_X(\psi(w)) \tag{2}$$

where $\psi = \phi^{-1}$. Therefore, the inverse function ψ must be known. In the case of the nonlinear power flow equations, such an analytical expression cannot be calculated. Therefore, it mat be useful to find a method for computing the statistics of W given the statistics (mean and variance) of the underlying distribution X. In order to solve this problem, we turn to the Delta Method.

B. Introduction to the Delta Method

First Order Approximation: The theory behind using the Delta Method to compute the variance of a function is simple. First, a function $g(X)$ is linearized via the Taylor Series approximation. The linearization point is μ_X, which is the mean value of the random variable X.

$$W = g(X) \approx g(\mu_X) + g'(\mu_X)(X - \mu_X) \tag{3}$$

Next, the variance operator is applied to both sides of the expression.

$$\begin{aligned} \mathrm{Var}(g(X)) &\approx \mathrm{Var}\left(g(\mu_X) + g'(\mu_X)(X - \mu_X)\right) \\ &\approx g'(\mu_X)^2 \mathrm{Var}(X) \end{aligned} \tag{4}$$

Therefore, we have that the variance of the random variable W is approximately equal to the variance of X weighted by the squared first derivative of g evaluated at the mean value of X.

$$\sigma_W^2 \approx g'(\mu_X)^2 \sigma_X^2 \tag{5}$$

Second Order Approximation: In order to calculate a more exact estimate of the variance of W, a higher order (second order) Taylor series approximation can be employed. Again, we assume $W = g(X)$.

$$\begin{aligned} g(X) &\approx g(\mu_X) + g'(\mu_X)(X - \mu_X) \\ &\quad + \frac{g''(\mu_X)}{2}(X - \mu_X)^2 \\ &\approx g(\mu_X) + g'(\mu_X)X - g'(\mu_X)\mu_X \\ &\quad + \frac{g''(\mu_X)}{2}\left(X^2 + \mu_X^2 - 2X\mu_X\right) \\ &\approx X\left(\left(\frac{g''(\mu_X)}{2}\right)X + \left(g'(\mu_X) - g''(\mu_X)\mu_X\right)\right) \\ &\quad + \left(g(\mu_X) + \frac{g''(\mu_X)}{2}\mu_X^2 - g'(\mu_X)\mu_X\right) \end{aligned}$$

Finally, we can take the variance of both sides.

$$\begin{aligned} \mathrm{Var}\left(g(X)\right) \approx \mathrm{Var} X &\left(\left(\left(\frac{g''(\mu_X)}{2}\right)X + \left(g'(\mu_X) - g''(\mu_X)\mu_X\right)\right)\right. \\ &\left. + \left(g(\mu_X) + \frac{g''(\mu_X)}{2}\mu_X^2 - g'(\mu_X)\mu_X\right)\right) \\ = \mathrm{Var}&\left(X\left(\left(\frac{g''(\mu_X)}{2}\right)X + \left(g'(\mu_X) - g''(\mu_X)\mu_X\right)\right)\right) \end{aligned}$$

The problem can now be reformulated.

$$\mathrm{Var}\left(X\left(\left(\frac{g''(\mu_X)}{2}\right)X + \left(g'(\mu_X) - g''(\mu_X)\mu_X\right)\right)\right) = \mathrm{Var}\left(X\left(aX + b\right)\right)$$

Where we have made the following substitutions:

$$a = \left(\frac{g''(\mu_X)}{2}\right)$$

$$b = \left(g'(\mu_X) - g''(\mu_X)\mu_X\right)$$

We must remember that the expectation operator is a linear operator:

$$\mathrm{E}((aX + b)) = a\mathrm{E}(X) + b$$

We can now compute this variance:

$$\begin{aligned} \mathrm{Var}\left(X\left(aX+b\right)\right) &= \mathrm{E}\left(\left(X\left(aX+b\right)\right)^2\right) - \mathrm{E}(X\left(aX+b\right))^2 \\ &= \mathrm{E}\left(X^2\left(a^2X^2 + b^2 + 2abX\right)\right) - \mathrm{E}(aX^2 + bX)^2 \\ &= \mathrm{E}\left(a^2X^4\right) + \mathrm{E}\left(b^2X^2\right) + \mathrm{E}\left(2abX^3\right) - \left(a\mathrm{E}\left(X^2\right) + b\mathrm{E}\left(X\right)\right)^2 \\ &= a^2\mathrm{E}\left(X^4\right) + b^2\mathrm{E}\left(X^2\right) + 2ab\mathrm{E}\left(X^3\right) - \left(a\mathrm{E}\left(X^2\right) + b\mathrm{E}\left(X\right)\right)^2 \end{aligned}$$

Each of these can now be computed. Initially, we only know the following values: $\mathrm{E}(X)$ and $\mathrm{Var}\left(X\right)$. Using just these values, all of the moments of X can be determined (see Wikipedia).

$$\mathrm{E}(X) = \mathrm{E}(X)$$

$$\mathrm{E}(X^2) = \mathrm{E}(X)^2 + \mathrm{Var}\left(X\right)$$

$$\mathrm{E}(X^3) = \mathrm{E}(X)^3 + 3\mathrm{E}(X)\mathrm{Var}\left(X\right)$$

$$\mathrm{E}(X^4) = \mathrm{E}(X)^4 + 6\mathrm{E}(X)^2\mathrm{Var}\left(X\right) + 3\mathrm{Var}\left(X\right)^2$$

This final expression for the variance is arrived at:

$$\begin{aligned} \mathrm{Var}\left(X\left(aX+b\right)\right) = a^2&\left(\mathrm{E}(X)^4 + 6\mathrm{E}(X)^2\mathrm{Var}\left(X\right) + 3\mathrm{Var}\left(X\right)^2\right) \\ &+ b^2\left(\mathrm{E}(X)^2 + \mathrm{Var}\left(X\right)\right) \\ &+ 2ab\left(\mathrm{E}(X)^3 + 3\mathrm{E}(X)\mathrm{Var}\left(X\right)\right) \\ &- \left(a\left(\mathrm{E}(X)^2 + \mathrm{Var}\left(X\right)\right) + b\mathrm{E}\left(X\right)\right)^2 \end{aligned} \tag{6}$$

C. Using the First and Second Order Delta Method to Quantify Load Noise

In order to show how load noise might be quantified, we need to apply the power flow equations to the simple two bus system shown in Figure 1.

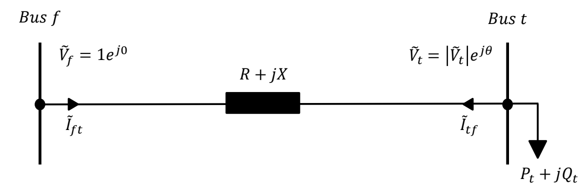

Figure 1. 2 Bus Power System with Shunt Elements Removed

The power flow equations (for a K bus system) are stated below:

$$P_i^{\mathrm{inj}} = \mathrm{V}_i \sum_{k=1}^{K} \mathrm{V}_k \left[G_{ik}\cos(\theta_{ik}) + B_{ik}\sin(\theta_{ik})\right] \quad i = 1, 2, ..., K \tag{7}$$

$$Q_i^{\mathrm{inj}} = \mathrm{V}_i \sum_{k=1}^{K} \mathrm{V}_k \left[G_{ik}\sin(\theta_{ik}) - B_{ik}\cos(\theta_{ik})\right] \quad i = 1, 2, ..., K \tag{8}$$

The 2×2 system Y-bus matrix is given.

$$Y = \begin{bmatrix} G + jB & -G - jB \\ -G - jB & G + jB \end{bmatrix} = \begin{bmatrix} Y_{ff} & Y_{ft} \\ Y_{tf} & Y_{tt} \end{bmatrix} \tag{9}$$

After applying the power flow equations and manipulating them in order to solve for the real and reactive load (algebra left out), we arrive at expressions (10) and (11).

$$P_t^{\mathrm{Load}} + \mathrm{V}_t^2 G = \mathrm{V}_f \mathrm{V}_t G\cos(\theta_t) + \mathrm{V}_f \mathrm{V}_t B\sin(\theta_t) \tag{10}$$

$$Q_t^{\mathrm{Load}} - \mathrm{V}_t^2 B = \mathrm{V}_f \mathrm{V}_t G\sin(\theta_t) - \mathrm{V}_f \mathrm{V}_t B\cos(\theta_t) \tag{11}$$

Also, we can write the complex power of the load in the following way, where $\beta = \tan(\phi)$ is a power factor parameter and ϕ is the phase shift between the load voltage and current $(\theta_v - \theta_i)$.

$$S_t^{\mathrm{Load}} = P_D(1 + j\beta)$$

Ultimately, we wish to solve for V_t explicitly. To eliminate the nonlinear trigonometric terms, we square both expressions (θ_t can be eliminated through the identity $\sin^2\theta + \cos^2\theta = 1$). We also write the active and reactive load demand in terms of P_D and β.

$$\left(P_D + \mathrm{V}_t^2 G\right)^2 = \left(\mathrm{V}_f \mathrm{V}_t G\cos(\theta_t) + \mathrm{V}_f \mathrm{V}_t B\sin(\theta_t)\right)^2 \tag{12}$$

$$\left(\beta P_D - \mathrm{V}_t^2 B\right)^2 = \left(\mathrm{V}_f \mathrm{V}_t G\sin(\theta_t) - \mathrm{V}_f \mathrm{V}_t B\cos(\theta_t)\right)^2 \tag{13}$$

Computing the square of each side yields the following expanded expressions.

$$\begin{aligned} &P_D^2 + \mathrm{V}_t^4 G^2 + 2P_D \mathrm{V}_t^2 G = \\ &\mathrm{V}_f{}^2 \mathrm{V}_t^2 G^2\cos^2(\theta_t) + \mathrm{V}_f{}^2 \mathrm{V}_t^2 B^2\sin^2(\theta_t) + 2\mathrm{V}_f{}^2 \mathrm{V}_t^2 BG\sin(\theta_t)\cos(\theta_t) \end{aligned}$$

$$\beta^2 P_D^2 + \mathrm{V}_t^4 B^2 - 2\beta P_D \mathrm{V}_t^2 B =$$
$$\mathrm{V}_f{}^2 \mathrm{V}_t^2 G^2 \sin^2(\theta_t) + \mathrm{V}_f{}^2 \mathrm{V}_t^2 B^2 \cos^2(\theta_t) - 2\mathrm{V}_f{}^2 \mathrm{V}_t^2 BG \sin(\theta_t)\cos(\theta_t)$$

These expressions can now be summed together.

$$P_D^2 + \mathrm{V}_t^4 G^2 + 2P_D \mathrm{V}_t^2 G + \beta^2 P_D^2 + \mathrm{V}_t^4 B^2 - 2\beta P_D \mathrm{V}_t^2 B =$$
$$\mathrm{V}_f{}^2 \mathrm{V}_t^2 G^2 \sin^2(\theta_t) + \mathrm{V}_f{}^2 \mathrm{V}_t^2 B^2 \cos^2(\theta_t) - 2\mathrm{V}_f{}^2 \mathrm{V}_t^2 BG \sin(\theta_t)\cos(\theta_t) +$$
$$\mathrm{V}_f{}^2 \mathrm{V}_t^2 G^2 \cos^2(\theta_t) + \mathrm{V}_f{}^2 \mathrm{V}_t^2 B^2 \sin^2(\theta_t) + 2\mathrm{V}_f{}^2 \mathrm{V}_t^2 BG \sin(\theta_t)\cos(\theta_t)$$

$$\Downarrow$$

$$P_D^2 + \mathrm{V}_t^4 G^2 + 2P_D \mathrm{V}_t^2 G + \beta^2 P_D^2 + \mathrm{V}_t^4 B^2 - 2\beta P_D \mathrm{V}_t^2 B =$$
$$\mathrm{V}_f{}^2 \mathrm{V}_t^2 G^2 \left(\sin^2(\theta_t) + \cos^2(\theta_t)\right) + \mathrm{V}_f{}^2 \mathrm{V}_t^2 B^2 \left(\cos^2(\theta_t) + \sin^2(\theta_t)\right)$$

$$\Downarrow$$

$$P_D^2 + \mathrm{V}_t^4 G^2 + 2P_D \mathrm{V}_t^2 G + \beta^2 P_D^2 + \mathrm{V}_t^4 B^2 - 2\beta P_D \mathrm{V}_t^2 B =$$
$$\mathrm{V}_f{}^2 \mathrm{V}_t^2 G^2 + \mathrm{V}_f{}^2 \mathrm{V}_t^2 B^2$$

$$\Downarrow$$

$$\begin{aligned} 0 \;=\;& \mathrm{V}_t^4\left(G^2 + B^2\right) + \mathrm{V}_t^2\left(2P_D G - 2\beta P_D B - \mathrm{V}_f{}^2 G^2 - \mathrm{V}_f{}^2 B^2\right) \\ &+ \mathrm{V}_t^0\left(P_D^2 + \beta^2 P_D^2\right) \end{aligned}$$

For simplicity of notation, we define the following variables:

$$a_1 = \left(G^2 + B^2\right)$$
$$a_2 = \left(2P_D G - 2\beta P_D B - \mathrm{V}_f{}^2 G^2 - \mathrm{V}_f{}^2 B^2\right)$$
$$a_3 = \left(P_D^2 + \beta^2 P_D^2\right)$$

After summing, simplifying, and applying the quadratic formula (math omitted), we obtain the following expression for the magnitude of the bus voltage:

$$\mathrm{V}_t = +\sqrt{\frac{-a_2 \pm \sqrt{a_2^2 - 4a_1 a_3}}{2a_1}} \tag{14}$$

Now, we have a function for load voltage. In the most idealistic world of Figure 1, the voltage at the from bus (potentially a generator bus) is held constant. The only independent stochastic variable of the load. Using the first order delta method (see equation (5)), solving for the load noise is trivial, since we have an expression for voltage which is a function of load. Therefore:

$$\sigma_V^2 \approx f_V'(\mu_P)^2 \sigma_P^2$$

We rearrange and solve for load noise:

$$\sigma_P^2 = \frac{\sigma_V^2}{f_V'(\mu_P)^2} \tag{15}$$

It shall now be shown how the expression derived in (6) can be used to quantify the noise of a load. In this expression, the variables a and b were defined in the following way:

$$a = \frac{g''(\mu_X)}{2}$$

$$b = g'(\mu_X) - g''(\mu_X)\mu_X$$

We can write these equations in terms of load power variance and bus voltage variance, where we have the function $V_t = g(P)$ (written using simplified notation). The base load value is μ_P and the variance of the load power is σ_P^2. We redefine a and b.

$$a = \frac{g''(\mu_P)}{2}$$

$$b = g'(\mu_P) - g''(\mu_P)\mu_P$$

With this clarification, we can move forward with the variance calculation.

$$\begin{aligned} \sigma_{V_t}^2 &= a^2\left(\mu_P^4 + 6\mu_P^2\sigma_P^2 + 3\sigma_P^4\right) + b^2\left(\mu_P^2 + \sigma_P^2\right) \\ &+ 2ab\left(\mu_P^3 + 3\mu_P\sigma_P^2\right) - \left(a\left(\mu_P^2 + \sigma_P^2\right) + b\mu_P\right)^2 \\ &= a^2\mu_P^4 + a^2 6\mu_P^2\sigma_P^2 + a^2 3\sigma_P^4 + b^2\mu_P^2 + b^2\sigma_P^2 \\ &+ 2ab\mu_P^3 + 6ab\mu_P\sigma_P^2 - \left(a\mu_P^2 + a\sigma_P^2 + b\mu_P\right)^2 \\ &= a^2\mu_P^4 + a^2 6\mu_P^2\sigma_P^2 + a^2 3\sigma_P^4 + b^2\mu_P^2 + b^2\sigma_P^2 \\ &+ 2ab\mu_P^3 + 6ab\mu_P\sigma_P^2 - a^2\mu_P^4 - a^2\sigma_P^4 - b^2\mu_P^2 \\ &- 2a^2\mu_P^2\sigma_P^2 - 2a\sigma_P^2 b\mu_P - 2ab\mu_P^3 \\ &= \left(2a^2\right)\sigma_P^4 + \left(a^2 6\mu_P^2 + b^2 + 6ab\mu_P - 2a^2\mu_P^2 - 2ab\mu_P\right)\sigma_P^2 \\ &\quad + \left(a^2\mu_P^4 + b^2\mu_P^2 + 2ab\mu_P^3 - a^2\mu_P^4 - b^2\mu_P^2 - 2ab\mu_P^3\right)\sigma_P^0 \end{aligned}$$

The voltage variance can be pulled to the other side.

$$\begin{aligned} 0 &= \left(2a^2\right)\sigma_P^4 + \left(a^2 6\mu_P^2 + b^2 + 6ab\mu_P - 2a^2\mu_P^2 - 2ab\mu_P\right)\sigma_P^2 \\ &+ \left(a^2\mu_P^4 + b^2\mu_P^2 + 2ab\mu_P^3 - a^2\mu_P^4 - b^2\mu_P^2 - 2ab\mu_P^3 - \sigma_{V_t}^2\right)\sigma_P^0 \\ &= \left(2a^2\right)\sigma_P^4 + \left(4a^2\mu_P^2 + b^2 + 4ab\mu_P\right)\sigma_P^2 + \left(-\sigma_{V_t}^2\right)\sigma_P^0 \end{aligned}$$

We employ the following symbols:

$$c_1 = \left(2a^2\right)$$

$$c_2 = \left(4a^2\mu_P^2 + b^2 + 4ab\mu_P\right)$$

$$c_3 = \left(-\sigma_V^2\right)$$

The quadratic formula can be used to solve for the variance of the load power.

$$\sigma_P^2 = \frac{-\left(c_2\right) \pm \sqrt{\left(c_2\right)^2 - 4c_1 c_2}}{2c_1} \tag{16}$$

D. Results: Testing the Derived Expressions to Estimate Load Noise.

In order test the validity of the expressions given by (15) and (16), a simple 2 bus system is simulated in the MATLAB toolbox Power System Analysis Toolbox (PSAT). For a variety of load levels, an analytical solver was used to compute the full covariance matrix for the entire system. After each solve, the expected mean and variance of the bus voltage and load demand were computed. In Figure (2), three quantities are contrasted: first order delta method load variance estimation, second order delta method load variance estimation, and expected load variance (if a time domain simulation was carried out until $t = \infty$, this the value that the load variance would converge to).

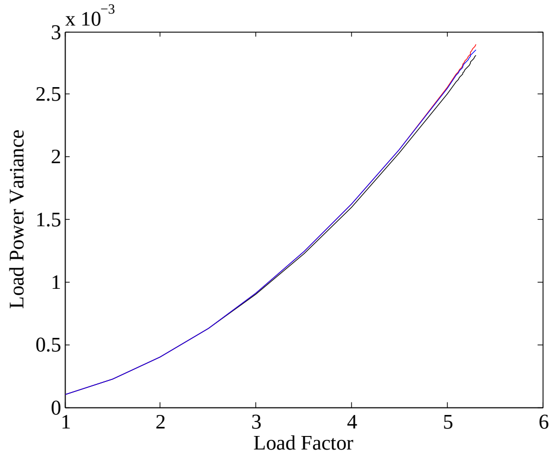

Figure 2. Load Noise Quantification. The top red line is the first order delta method estimation (see equation (15)), the middle blue line is the second order delta method estimation (see equation (16)), and the bottom black line is the true load variance.

As can be seen in Figure 2, the first and second order delta methods yield almost identical approximations. Most likely, higher order approximations would not improve the first order case drastically, so moving forward, the second order method will not be used.

Clearly, there is some sort of error between the load variance approximation and the true load variance. It is assumed that higher order approximations will not improve upon this error, given the discussion in the previous paragraph. Therefore, there is some sort of inherent error in the model or process which is being used. Initially, it was assumed that the error was based in the fact that equations (12) and (13) are both functions of θ while derived equation (14) is not. The θ variable can be thought of as a random variable, so eliminating from the calculation of bus voltage variance might cause problems. After investigating this lead, though, it turns out that this is not the case: θ can be eliminating with compromising the integrity of the delta method.

The true cause of the error, in fact, caused by the parameter β

III. Quantifying Voltage Stability in a fully Observable System

A. Introduction to the Bus Voltage Deviation Index

The goal of the following sections is to introduce a method which will have the ability to generate a voltage stability index for every node in an entire system. Ultimately, each index is based on computing a ratio of the real time variance and a "critical variance" value. The novel Bus Voltage Deviation Index is defined like so.

$$\mathrm{BVDI} = \frac{\sigma_V(t)}{\sigma_{V-critical}} \tag{17}$$

The following method depends on the fact that the system is fully observable. In other words, the "full color" problem has been solved, and all voltage and current phasors throughout the system are known in real time. Not only is phasor data known, but so is real time (numerical) variance, covariance, and correlation for all system voltages and currents. The numerical and variance and covariance for given vectors X and Y can be computed from time series data like so:

$$\mathrm{Cov}(X,Y) = \frac{1}{N-1}\sum (x-\mu_x)(y-\mu_y) \tag{18}$$

$$\mathrm{Var}(X) = \frac{1}{N-1}\sum (x-\mu_x)^2 \tag{19}$$

B. Delta Method for the Multivariate Taylor Series

It will now be shown how the first order multivariate Taylor series can be used with the first order delta method in order to compute the critical variance, $\sigma_{V-critical}$. First, Taylor Series approximation is used on the function $f(x,y,z)$ at the point (a,b,c). To be clear, this process can be extended to a function which is dependent upon many variables. This will be necessary below. For the three variable case, we have the following:

$$\begin{aligned} f(x,y,z) &\approx f(a,b,c) + \frac{\partial f(x,y,z)}{\partial x}|_{a,b,c}(x-a) \\ &+ \frac{\partial f(x,y,z)}{\partial y}|_{a,b,c}(y-b) + \frac{\partial f(x,y,z)}{\partial z}|_{a,b,c}(z-c) \\ &= f(a,b,c) + x\frac{\partial f(x,y,z)}{\partial x}|_{a,b,c} - a\frac{\partial f(x,y,z)}{\partial x}|_{a,b,c} \\ &+ y\frac{\partial f(x,y,z)}{\partial y}|_{a,b,c} - b\frac{\partial f(x,y,z)}{\partial y}|_{a,b,c} \\ &+ z\frac{\partial f(x,y,z)}{\partial z}|_{a,b,c} - c\frac{\partial f(x,y,z)}{\partial z}|_{a,b,c} \end{aligned} \tag{20}$$

Now, the delta method can be applied in order to estimate the variance of $f(x,y,z)$. In this case, we model $X = x$, $Y = y$, and $Z = z$ all as random variables (Gaussians). After making this substitution, we have the following.

$$\begin{aligned} \mathrm{Var}(f(x,y,z)) \approx \mathrm{Var}\Big(& f(a,b,c) + X\frac{\partial f(x,y,z)}{\partial x}|_{a,b,c} \\ & -a\frac{\partial f(x,y,z)}{\partial x}|_{a,b,c} + Y\frac{\partial f(x,y,z)}{\partial y}|_{a,b,c} - b\frac{\partial f(x,y,z)}{\partial y}|_{a,b,c} \\ & +Z\frac{\partial f(x,y,z)}{\partial z}|_{a,b,c} - c\frac{\partial f(x,y,z)}{\partial z}|_{a,b,c}\Big) \end{aligned}$$

For convenience, we define a new random variable:

$$\begin{aligned} W = & X\frac{\partial f(x,y,z)}{\partial x}|_{a,b,c} + Y\frac{\partial f(x,y,z)}{\partial y}|_{a,b,c} + Z\frac{\partial f(x,y,z)}{\partial z}|_{a,b,c} \\ & + f(a,b,c) - a\frac{\partial f(x,y,z)}{\partial x}|_{a,b,c} - b\frac{\partial f(x,y,z)}{\partial y}|_{a,b,c} \\ & - c\frac{\partial f(x,y,z)}{\partial z}|_{a,b,c} \end{aligned}$$

This new random variable will have the following mean and variance values.

$$\begin{aligned} \mathrm{E}[W] = & \mathrm{E}[X]\frac{\partial f(x,y,z)}{\partial x}|_{a,b,c} + \mathrm{E}[Y]\frac{\partial f(x,y,z)}{\partial y}|_{a,b,c} \\ & + \mathrm{E}[Z]\frac{\partial f(x,y,z)}{\partial z}|_{a,b,c} + f(a,b,c) - a\frac{\partial f(x,y,z)}{\partial x}|_{a,b,c} \\ & - b\frac{\partial f(x,y,z)}{\partial y}|_{a,b,c} - c\frac{\partial f(x,y,z)}{\partial z}|_{a,b,c} \end{aligned}$$

$$\begin{aligned}\mathrm{Var}(W) &= \left(\frac{\partial f(x,y,z)}{\partial x}|_{a,b,c}\right)^2 \mathrm{Var}(X) + \left(\frac{\partial f(x,y,z)}{\partial y}|_{a,b,c}\right)^2 \mathrm{Var}(Y)\\ &+ \left(\frac{\partial f(x,y,z)}{\partial z}|_{a,b,c}\right)^2 \mathrm{Var}(Z)\\ &+ 2\left(\frac{\partial f(x,y,z)}{\partial x}|_{a,b,c}\right)\left(\frac{\partial f(x,y,z)}{\partial y}|_{a,b,c}\right)\mathrm{Cov}(X,Y)\\ &+ 2\left(\frac{\partial f(x,y,z)}{\partial y}|_{a,b,c}\right)\left(\frac{\partial f(x,y,z)}{\partial z}|_{a,b,c}\right)\mathrm{Cov}(Y,Z)\\ &+ 2\left(\frac{\partial f(x,y,z)}{\partial z}|_{a,b,c}\right)\left(\frac{\partial f(x,y,z)}{\partial x}|_{a,b,c}\right)\mathrm{Cov}(Z,X)\end{aligned} \tag{21}$$

This can now be related to a power system. For a simple 2 bus system, we assume that the voltage at a bus is a function of three types of dynamic variables: (1) load demand (2) surrounding bus voltages, and (3) phase angle differences (system topology is assumed static). Therefore, the following function exists: $V_t = f(V_f, P_d, \theta_{tf})$. Ultimately, the multivariate delta method is what will be used to compute the critical variance (based on a critical loading level).

C. A function for Bus Voltage based on the Power Flow Equations (Real Power)

Since we are interested in applying this method (separately) to all buses in a system, we must deal with buses which may have many interconnections. Therefore, the voltage at a given bus is a function of many other bus voltages. In order to write bus voltage as a function of other bus voltages, we employ the power flow equations (7) and (8). After some algebraic manipulation, we arrive at two function for real and reactive power demand at bus i. It is assumed that bus i (the bus under consideration) is connected to bus 1, bus n, and a few buses in between.

$$\begin{aligned}P_i^{\mathrm{Load}} &= -\mathrm{V}_i^2 G_{i,i} - \mathrm{V}_i\mathrm{V}_1\left(G_{i,1}\cos(\theta_{i,1}) + B_{i,1}\sin(\theta_{i,1})\right) - \cdots\\ &- \mathrm{V}_i\mathrm{V}_n\left(G_{i,n}\cos(\theta_{i,n}) + B_{i,n}\sin(\theta_{i,n})\right)\end{aligned}$$

$$\begin{aligned}Q_i^{\mathrm{Load}} &= \mathrm{V}_i^2 B_{i,i} + \mathrm{V}_i\mathrm{V}_1\left(B_{i,1}\cos(\theta_{i,1}) - G_{i,1}\sin(\theta_{i,1})\right) + \cdots\\ &+ \mathrm{V}_i\mathrm{V}_n\left(B_{i,n}\cos(\theta_{i,n}) + G_{i,n}\sin(\theta_{i,n})\right)\end{aligned}$$

In this case, the function for Q_i^{Load} does not give us any other additional information about the i^{th} bus voltage, so we neglect it. We only need one function for V_i to implement the delta method. We now rewrite the expression above and group terms together.

$$\begin{aligned}0 &= (-G_{ii})\,\mathrm{V}_i^2 + -\mathrm{V}_1\left(G_{i,1}\cos(\theta_{i,1}) + B_{i,1}\sin(\theta_{i,1})\right)\mathrm{V}_i^1 -\\ &\cdots - \mathrm{V}_n\left(G_{i,n}\cos(\theta_{i,n}) + B_{i,n}\sin(\theta_{i,n})\right)\mathrm{V}_i^1 + (-P_D)\,\mathrm{V}_i^0\end{aligned}$$

$$\Downarrow$$

$$0 = a\mathrm{V}_i^2 + b\mathrm{V}_i^1 + c\mathrm{V}_i^0$$

Once again, we are assuming full observability of the system. Therefore, all line parameters, all phase angles, and all voltages are known values. We can then compute the actual values of a, b, and c above, and using the quadratic formula, V_i can be calculated.

$$\mathrm{V}_i = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{22}$$

We only care about the negative value of the quadratic formula (as verified in MATLAB), so (22) can be simplified.

$$\mathrm{V}_i = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{23}$$

D. Implementing the Delta Method on the Bus Voltage Function

In a power system, we wish to model the i^{th} bus power demand and all bus voltages as random variables. For simplicity, let's assume bus i is connected to one bus: bus 1. We then have the following equation for V_i.

$$\begin{aligned}\mathrm{V}_i = f\left(\mathrm{V}_1, P_D, \theta_{i,1}\right) &= \frac{\mathrm{V}_1\left(G_{i,1}\cos(\theta_{i,1}) + B_{i,1}\sin(\theta_{i,1})\right)}{2\left(-G_{ii}\right)}\\ &- \frac{\sqrt{\left(\mathrm{V}_1\left(G_{i,1}\cos(\theta_{i,1}) + B_{i,1}\sin(\theta_{i,1})\right)\right)^2 - 4\left(-G_{ii}\right)\left(-P_D\right)}}{2\left(-G_{ii}\right)}\end{aligned}$$

Now, we apply Taylor series, where we have the following derivatives:

$$\begin{aligned}\frac{\partial f\left(\mathrm{V}_1, P_D, \theta_{i,1}\right)}{\partial \mathrm{V}_1} &= \frac{s_1 - \frac{1}{2}\left(\mathrm{V}_1\left(s_1\right)\right)^2}{2\left(-G_{ii}\right)}\\ &\frac{-4\left(-G_{ii}\right)\left(-P_D\right)^{-\frac{1}{2}}2\left(\mathrm{V}_1 s_1\right)\left(s_1\right)}{2\left(-G_{ii}\right)}\end{aligned}$$

$$\begin{aligned}\frac{\partial f\left(\mathrm{V}_1, P_D, \theta_{i,1}\right)}{\partial P_D} &=\\ &\frac{-\frac{1}{2}\left(\left(\mathrm{V}_1 s_1\right)^2 - 4\left(G_{ii}\right)\left(P_D\right)\right)^{-\frac{1}{2}}\left(-4G_{ii}\right)}{2\left(-G_{ii}\right)}\end{aligned}$$

$$\begin{aligned}\frac{\partial f\left(\mathrm{V}_1, P_D, \theta_{i,1}\right)}{\partial \theta_{i,1}} &= \frac{\mathrm{V}_1\left(-G_{i,1}\sin(\theta_{i,1}) + B_{i,1}\cos(\theta_{i,1})\right)}{2\left(-G_{ii}\right)}\\ &- \frac{\frac{1}{2}\left(\left(\mathrm{V}_1\left(s_1\right)\right)^2 - 4\left(-G_{ii}\right)\left(-P_D\right)\right)^{-\frac{1}{2}}\left(2\mathrm{V}_1\left(s_1\right)\right)}{2\left(-G_{ii}\right)}\\ &\times\left(-G_{i,1}\sin(\theta_{i,1}) + B_{i,1}\cos(\theta_{i,1})\right)\end{aligned}$$

Where we have made the following substitution for notation's sake:

$$s_1 = G_{i,1}\cos(\theta_{i,1}) + B_{i,1}\sin(\theta_{i,1})$$

Now that the derivatives are all computed, we can compute the variance of V_i with relative ease. Equation (21) is leveraged from above, and we evaluate the derivatives at the mean values of V_i, P_D, and $\theta_{i,1}$. We define the following.

$$a = \mu_{\mathrm{V}_i}$$
$$b = \mu_{P_D}$$
$$c = \mu_{\theta_{i,1}}$$

Based on these definitions, we have the following expression for the variance of the bus voltage.

$$\begin{aligned}\mathrm{Var}(\mathrm{V}_i) &= \left(\frac{\partial f\left(\mathrm{V}_1, P_D, \theta_{i,1}\right)}{\partial \mathrm{V}_1}|_{a,b,c}\right)^2 \mathrm{Var}(\mathrm{V}_i)\\ &+ \left(\frac{\partial f\left(\mathrm{V}_1, P_D, \theta_{i,1}\right)}{\partial P_D}|_{a,b,c}\right)^2 \mathrm{Var}(P_D)\\ &+ \left(\frac{\partial f\left(\mathrm{V}_1, P_D, \theta_{i,1}\right)}{\partial \theta_{i,1}}|_{a,b,c}\right)^2 \mathrm{Var}(\theta_{i,1})\\ &+2\left(\frac{\partial f\left(\mathrm{V}_1, P_D, \theta_{i,1}\right)}{\partial \mathrm{V}_1}|_{a,b,c}\right)\left(\frac{\partial f\left(\mathrm{V}_1, P_D, \theta_{i,1}\right)}{\partial P_D}|_{a,b,c}\right)\mathrm{Cov}(\mathrm{V}_i, P_D)\\ &+2\left(\frac{\partial f\left(\mathrm{V}_1, P_D, \theta_{i,1}\right)}{\partial P_D}|_{a,b,c}\right)\left(\frac{\partial f\left(\mathrm{V}_1, P_D, \theta_{i,1}\right)}{\partial \theta_{i,1}}|_{a,b,c}\right)\mathrm{Cov}(P_D, \theta_{i,1})\\ &+2\left(\frac{\partial f\left(\mathrm{V}_1, P_D, \theta_{i,1}\right)}{\partial \theta_{i,1}}|_{a,b,c}\right)\left(\frac{\partial f\left(\mathrm{V}_1, P_D, \theta_{i,1}\right)}{\partial \mathrm{V}_1}|_{a,b,c}\right)\mathrm{Cov}(\theta_{i,1}, \mathrm{V}_i)\end{aligned} \tag{24}$$

There are, therefore, nine statistical values which must be know: three mean values, three variance values, and three covariance values. The mean values are computed like so:

$$\mu_{V_i} = \frac{1}{N}\sum V_i$$

$$\mu_{P_D} = \frac{1}{N}\sum \Re\left(\left(\tilde{V}_{load}\right)\left(\tilde{I}_{load}\right)^*\right)$$

$$\mu_{\theta_{i,1}} = \frac{1}{N}\sum \theta_i - \theta_1$$

The variances can also be computed numerically.

$$\sigma^2_{V_1} = \frac{1}{N-1}\sum \left(V_i - \mu_{V_i}\right)^2$$

$$\sigma^2_{P_D} = \frac{1}{N-1}\sum \left(\Re\left(\left(\tilde{V}_{load}\right)\left(\tilde{I}_{load}\right)^*\right) - \mu_{P_D}\right)^2$$

$$\sigma^2_{\theta_{i,1}} = \frac{1}{N-1}\sum \left((\theta_i - \theta_1) - \mu_{\theta_{i,1}}\right)^2$$

The covariance values are computed in similar ways.

$$\sigma_{V_1,P_D} = \frac{1}{N-1}\sum \left(V_i - \mu_{V_i}\right)\left(\Re\left(\left(\tilde{V}_{load}\right)\left(\tilde{I}_{load}\right)^*\right) - \mu_{P_D}\right)$$

$$\sigma_{P_D,\theta_{i,1}} = \frac{1}{N-1}\sum \left(\Re\left(\left(\tilde{V}_{load}\right)\left(\tilde{I}_{load}\right)^*\right) - \mu_{P_D}\right)\left((\theta_i - \theta_1) - \mu_{\theta_{i,1}}\right)$$

$$\sigma_{\theta_{i,1},V_1} = \frac{1}{N-1}\sum \left((\theta_i - \theta_1) - \mu_{\theta_{i,1}}\right)\left(V_i - \mu_{V_i}\right)$$

E. 4 Bus System Numerical Validation

In order to test this method, it is applied to the final bus on the end of a 4-bus radial network (bus L_3 in the following "diagram"). **The goal of this test is to show that we can compute (predict) the variance of the voltage at a bus given raw PMU data for all quantities except voltage variance at bus L_3.** As will be shown, this will allow us to eventually compute the "critical variance" $\sigma_{V-critical}$.

$$G - - - L_1 - - - L_2 - - - L_3$$

Each PQ bus feeds a single load. Of course, bus L_3 only has one transmission line connecting to it, so it is a very simple case. The procedure outlined in the previous section was followed exactly in order to predict the variance of the bus voltage. As the system is increasingly loaded, equation (24) is used after running a 50s time domain simulation. Results show the ability of this method to predict the bus voltage magnitude variance. These results are shown in Figure 3.

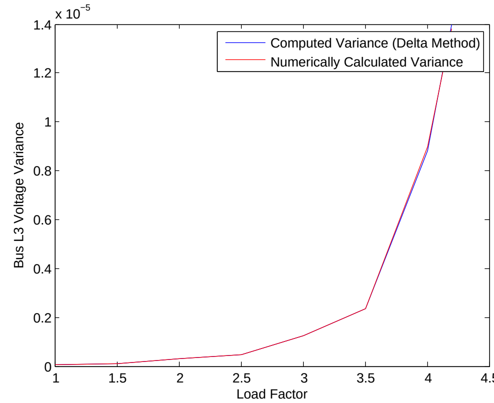

Figure 3. Computed vs Calculated Bus Voltage Variance

F. 39 Bus System Numerical Validation

In order to further validate the method, we test it on Bus 16 of the 39 bus system. The system is shown in Figure 4. Bus 16 has a load and is connected to 5 transmission lines, meaning it is a highly interconnected bus.

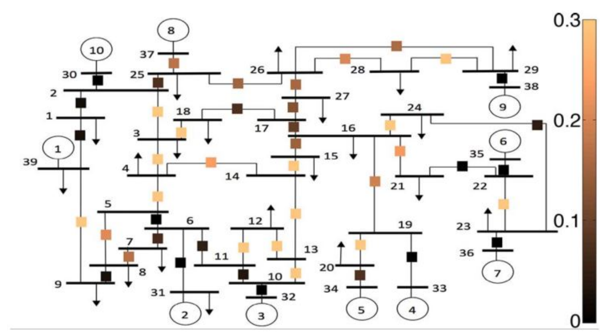

Figure 4. Topology of the 39 Bus System (Ignore the Heat Map Index)

To complete this task, the power flow equations must be applied to the system. As explained above, we can get away with just applying the real power flow equation (the reactive power equation can be ignored). Bus 16 has transmission lines connecting to buses 15, 17, 19, 21, and 24.

$$P_i^{\text{Load}} = -\left(P_i^{\text{inj}}\right) = -V_i\sum_{k=1}^{K} V_k\left[G_{ik}\cos(\theta_{ik}) + B_{ik}\sin(\theta_{ik})\right]$$

$$\Downarrow$$

$$\begin{aligned} P_{16}^{\text{Load}} = &-V_{16}^2 G_{16,16} \\ &- V_{16}V_{15}\left[G_{16,15}\cos(\theta_{16,15}) + B_{16,15}\sin(\theta_{16,15})\right] \\ &- V_{16}V_{17}\left[G_{16,17}\cos(\theta_{16,17}) + B_{16,17}\sin(\theta_{16,17})\right] \\ &- V_{16}V_{19}\left[G_{16,19}\cos(\theta_{16,19}) + B_{16,19}\sin(\theta_{16,19})\right] \\ &- V_{16}V_{21}\left[G_{16,21}\cos(\theta_{16,21}) + B_{16,21}\sin(\theta_{16,21})\right] \\ &- V_{16}V_{24}\left[G_{16,24}\cos(\theta_{16,24}) + B_{16,24}\sin(\theta_{16,24})\right] \end{aligned}$$

We simplify the notation and then group terms.

$$\begin{aligned} P_{16}^{\text{Load}} &= -\mathrm{V}_{16}^2 G_{16,16} - \mathrm{V}_{16} a - \mathrm{V}_{16} b - \mathrm{V}_{16} c - \mathrm{V}_{16} d - \mathrm{V}_{16} e \\ &= \mathrm{V}_{16}^2 \left(-G_{16,16}\right) - \mathrm{V}_{16} \left(a+b+c+d+e\right) \end{aligned}$$

Finally, we write the expression in a way which can be solved by the quadratic formula.

$$0 = \mathrm{V}_{16}^2 \left(-G_{16,16}\right) + \mathrm{V}_{16} \left(-\left(a+b+c+d+e\right)\right) + \left(-P_{16}^{\text{Load}}\right)$$

This can be solved explicitly.

$$\mathrm{V}_i = \frac{\left(-\left(a+\cdots+e\right)\right) - \sqrt{\left(-\left(a+\cdots+e\right)\right)^2 - 4\left(-G_{16,16}\right)\left(-P_{16}^{\text{Load}}\right)}}{2\left(-G_{16,16}\right)} \tag{25}$$

In order to prove that this complicated mess is a valid function for bus voltage, it is tested for a given load profile after running a power flow routine. The vector DAE.y contains PSAT's power flow results (θ, V), while the mathematical expression is the derived function for bus voltage. Clearly, both voltages are almost exactly the same. Therefore, equation (25) is a valid function for computing bus voltage.

$$\begin{aligned} \mathrm{DAE.y}(39+16) &= 1.03228 \\ \frac{\left(-(a+b+c+d+e)\right)-\sqrt{\left(-(a+b+c+d+e)\right)^2-4\left(-G_{16,16}\right)\left(-P_{16}^{\text{Load}}\right)}}{2\left(-G_{16,16}\right)} &= 1.03228 \end{aligned}$$

Now that the function has been written and tested, it must be transformed in two ways: (1) through the multivariate first order Taylor Series and then (2) through the delta method. Although a daunting task to solve the rest of this by hand, MATLAB's symbolic toolbox can be used to take all of the required derivatives. Because there are 11 random variables at play (5 bus voltages, 5 phase angle differences, and the bus load), the variance at bus 16 is the sum of 11 weighted variances and 55 weighted covariances:

$$\begin{aligned} \mathrm{Var}(\mathrm{V}_{16}) =& \sum_{i=1}^{11} \left(\frac{\partial \mathrm{V}_{16}}{\partial X_i}\right)^2 \mathrm{Var}(\mathrm{X}_i) \\ &+ \sum_{i=1}^{10} \left[\sum_{j=i+1}^{11} 2\left(\frac{\partial \mathrm{V}_{16}}{\partial X_i}\right)\left(\frac{\partial \mathrm{V}_{16}}{\partial X_j}\right) \mathrm{Cov}(\mathrm{X}_i, X_j)\right] \end{aligned}$$

The results for this method are given in Figure 5. The difference between the Delta Method Results and Numerically Calculated Results is very small, almost indistinguishable, for each load factor.

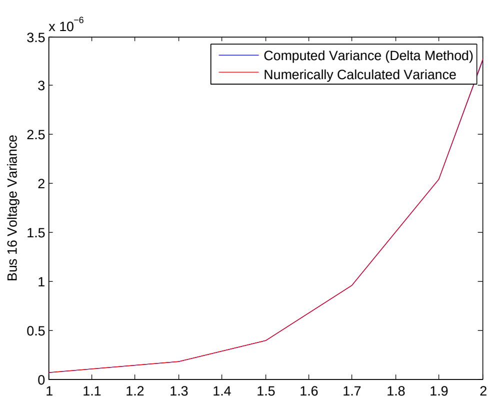

Figure 5. Computed vs Calculated Bus Voltage Variance

G. A Function for Bus Voltage based on the Power Flow Equations (Reactive Power)

Just as phase angle and real power flow are highly coupled, so are voltage and reactive power. Therefore, voltage collapse is highly related to reactive power shortages. In the previous sections, we depended on a function which relates bus voltage to real power. This function can just as easily be written in terms of reactive power. We begin with the equation developed for reactive power demand for a bus connected to n other buses.

$$\begin{aligned} Q_i^{\text{Load}} =& \mathrm{V}_i^2 B_{i,i} + \mathrm{V}_i \mathrm{V}_1 \left(B_{i,1}\cos(\theta_{i,1}) - G_{i,1}\sin(\theta_{i,1})\right) \\ &+ \cdots + \mathrm{V}_i \mathrm{V}_n \left(B_{i,n}\cos(\theta_{i,n}) - G_{i,n}\sin(\theta_{i,n})\right) \end{aligned}$$

We now write the function in a way that the quadratic equation can solve:

$$\begin{aligned} 0 =& \left(B_{ii}\right)\mathrm{V}_i^2 + \mathrm{V}_1\left(B_{i,1}\cos(\theta_{i,1}) - G_{i,1}\sin(\theta_{i,1})\right)\mathrm{V}_i^1 \\ &+ \cdots + \mathrm{V}_n\left(B_{i,n}\cos(\theta_{i,n}) - G_{i,1}\sin(\theta_{i,n})\right)\mathrm{V}_i^1 + \left(-Q_i^{\text{Load}}\right)\mathrm{V}_i^0 \end{aligned} \tag{26}$$

$$\Downarrow$$

$$0 = a\mathrm{V}_i^2 + b\mathrm{V}_i^1 + c\mathrm{V}_i^0$$

$$\Downarrow$$

$$\mathrm{V}_i = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{27}$$

At this point, the Taylor Series and then delta method can be applied, and the variance of the bus voltage $\sigma_{\mathrm{V}_i}^2$ can be solved for very accurately. The error margins are similar to those shown in Figures 1 and 3 (these experiments used load power P_D, rather than Q^{Load}, to solve for bus voltage variance).

H. Generating a Critical Variance Value

Now that these methods of bus voltage variance prediction have been developed, we can use them to predict a critical variance which should not be exceeded. This critical variance, clearly, is a function of many, many expected values, variances, and covariances, and it will therefore need to be constantly updated and recomputed. Therefore, the BVDI should be considered a dynamic index which will only have a relative meaning. Each time the index is recomputed, based on a dynamic critical variance and a real time measured variance, the following questions is answered: How close is the **current** bus voltage variance to the **current** critical variance?

$$\mathrm{BVDI} = \frac{\sigma_V(t)}{\sigma_{V-critical}}$$

As will be explained, the computed value for $\sigma_{V-critical}$ will be a conservative upper bound (i.e. the true critical variance will be higher). When a bus or load pocket approaches voltage collapse, it also approaches the point of maximum power transfer. This will be the point when Equation (27) only produces complex solutions. In other words, the radicand becomes negative because the c value becomes too large (c corresponds to P_D or Q_D, whichever is being used). Therefore, in order to solve for the critical variance value, the maximum transferable reactive power value $Q_{\max}$ must first be computed.

$$Q_{\max} = \frac{4a}{b^2}$$

The values of a and b can be inferred from equation (26). Solving for and using this maximum value of reactive power makes several bold assumptions.

1) This is a maximum for a given operating paradigm.
2) It assumes that phase angle differences do not change drastically for an increase in reactive power demand (a safe assumption until the system is at the voltage collapse threshold).
3) It also assumes that the voltages of surrounding buses do not change. This could be a safe assumption under certain conditions, but if there is an entire load pocket which is approaching voltage instability, then the computed value for $Q_{\max}$ for a given bus might be too liberal.
4) Finally, this assumption is based on the fact that $-Q_{\text{inj}}$ increases to $Q_{\max}$ while $-P_{\text{inj}}$ remains fixed. This, of course, is an impossible assumption, but it can be an acceptable one for the following reason: voltage collapse is very highly tied to reactive power flows. Real power flows contribute but in a much more minor sense. Increasing reactive power demand will still cause an increase in current (causing voltage drop across the real and imaginary components of the line impedance $R + jX$). Since we only care about computing $Q_{\max}$ in order to compute a critical voltage variance, which we find close to the voltage collapse, it may not be so important that we are assuming the $\Re(\tilde{S}_{\text{Load}})$ remains constant. Voltage collapse is voltage collapse.
5) We assume that all of the second order statistics remain fixed. This, once again, may not be a perfect assumption, but it will only make the critical voltage variance computation more conservative.

If these assumptions are too severe, a power flow could be run in order to recompute the voltages and phase angles of the surrounding buses. This would be a very time consuming option, and it is not realistic. It is important to remember, though, that the value for $\sigma_{V-critical}$ should be recomputed often. That means these assumptions will not be so severe.

Once the maximum reactive power demand has been identified and computed, a critical reactive power demand can be computed. In "Preventing Voltage Collapse with Protection Systems that Incorporate Optimal Reactive Power Control", Ajjarapu suggests that loading should not increase past 90% of critical loading. Therefore, we define the critical reactive power demand in the following way:

$$Q_{\text{critical}} = 0.9 \times Q_{\max}$$

Now that the critical reactive power demand has been identified, we can solve for the critical variance. An example for how to do so can be shown from the 4 bus system. The analytical expression for bus voltage variance (which is a function of only three random variables) simply needs to be evaluated at Q_{critical}, where $f\left(\mathrm{V}_1, Q_{\mathrm{D}}, \theta_{i,1}\right) = f$:

$$\begin{aligned}\sigma_{V-critical} = &\left(\frac{\partial f}{\partial \mathrm{V}_1}|_{a,b,c}\right)^2 \mathrm{Var}(\mathrm{V}_i) + \left(\frac{\partial f}{\partial P_D}|_{a,b,c}\right)^2 \mathrm{Var}(Q_{\mathrm{D}}) \\ &+ \left(\frac{\partial f}{\partial \theta_{i,1}}|_{a,b,c}\right)^2 \mathrm{Var}(\theta_{i,1}) + 2\left(\frac{\partial f}{\partial \mathrm{V}_1}|_{a,b,c}\right)\left(\frac{\partial f}{\partial P_D}|_{a,b,c}\right)\mathrm{Cov}(\mathrm{V}_i, Q_{\mathrm{D}}) \\ &+2\left(\frac{\partial f}{\partial P_D}|_{a,b,c}\right)\left(\frac{\partial f}{\partial \theta_{i,1}}|_{a,b,c}\right)\mathrm{Cov}(Q_{\mathrm{D}}, \theta_{i,1}) \\ &+2\left(\frac{\partial f}{\partial \theta_{i,1}}|_{a,b,c}\right)\left(\frac{\partial f}{\partial \mathrm{V}_1}|_{a,b,c}\right)\mathrm{Cov}(\theta_{i,1}, \mathrm{V}_i)\end{aligned} \tag{28}$$

Where the the function is linearized around the following points:

$$a = \mu_{\mathrm{V}_1}$$

$$b = Q_{\text{critical}}$$

$$c = \mu_{\theta_{i,1}}$$

In theory, this procedure can be performed on every bus in a system in real time (so long as we are dealing with a fully observable system). The system's overall voltage health is determined by the largest BVDI.

IV. Conclusion

In this project, the Delta Method was exploited in order to achieve two goals. First, an expression for quantifying load noise (variance) was developed. This method was tested using the first and second order Taylor Series approximations. For a simple two bus power system, with a fixed generator voltage, the method was shown to be highly accurate (within measurement error). Second, a method for computing the bus voltage variance of a bus in a highly interconnected system was determined. As explained previously, this method can then be used in order to compute a critical bus voltage variance. Grid operators may be able to use this critical voltage variance to determine when the system is becoming dangerously unstable.

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