Long Transmission Line Model

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Guiding Question: How do we convert the distributed parameters of a transmission line into lumped parameters?

Distributed Transmission Line Model

The following derivations are based off work by **James McCalley** and from ***Power System Dynamics: Stability and Control***. We consider the Transmission Line model given by Figure 1. In the diagram, all voltages and currents are sinusoidal phasor quantities oscillating at $\omega = \omega_0$. We have voltage $V = \mathrm{V}e^{\theta}$, current $I = \mathrm{I}e^{j\phi}$, impedance $z = \mathrm{z}e^{j\sigma}$ $(\frac{\Omega}{\mathrm{m}})$ and admittance $y = \mathrm{y}e^{j\psi}$ $(\frac{\Omega^{-1}}{\mathrm{m}})$.

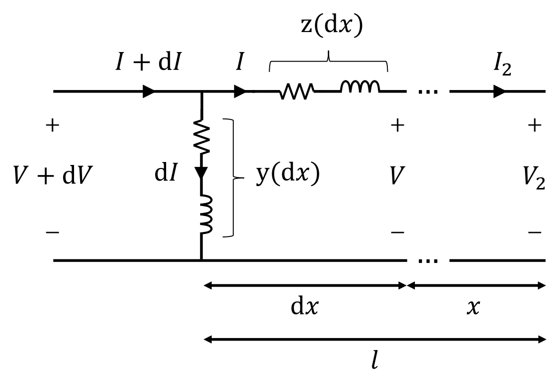

Figure 1: Single-phase transmission line equivalent circuit with *distributed* parameters

If we apply Kirchhoff's Voltage Law (KVL) around the outer loop, we have the following expression.

$$\begin{aligned} V + \mathrm{d}V - Iz\mathrm{d}x &= V \\ \mathrm{d}V &= Iz\mathrm{d}x \end{aligned} \tag{1}$$

We may calculate the change in voltage over a given change in distance.

$$\frac{\mathrm{d}V}{\mathrm{d}x} = Iz \tag{2}$$

We now apply Kirchhoff's Current Law (KCL) to the middle node.

$$\begin{aligned} (I + \mathrm{d}I) - (V + \mathrm{d}V)\, y\mathrm{d}x &= I \\ \mathrm{d}I &= (V + \mathrm{d}V)\, y\mathrm{d}x \end{aligned} \tag{3}$$

We may calculate the change in current over a given change in distance.

$$\begin{aligned} \frac{\mathrm{d}I}{\mathrm{d}x} &= (V + \mathrm{d}V)\, y \\ &\approx Vy \end{aligned} \tag{4}$$

We make the approximation that $\mathrm{d}V \ll V$, which is true when limits are taken. If we differentiate the expression for current change, we have the following useful result.

$$\frac{\mathrm{d}^2 I}{\mathrm{d}x^2} = \frac{\mathrm{d}V}{\mathrm{d}x} y \tag{5}$$

We now substitute in our expression for $\frac{\mathrm{d}V}{\mathrm{d}x}$ which was derived using KVL.

$$\frac{\mathrm{d}^2 I}{\mathrm{d}x^2} = Izy \tag{6}$$

At this point, the propagation constant $\gamma = \sqrt{zy}$ $\left(\frac{1}{\mathrm{m}}\right)$ may be invoked. From wikipedia: *The propagation constant of a sinusoidal electromagnetic wave is a measure of the change undergone by the amplitude and phase of the wave as it propagates in a given direction.* We now update the current phasor double derivative.

$$\frac{\mathrm{d}^2 I}{\mathrm{d}x^2} = I\gamma^2 \tag{7}$$

We now differentiate equation (2) with respect to x and then substitute in equation (4).

$$\begin{aligned}\frac{\mathrm{d}^2 V}{\mathrm{d}x^2} &= \frac{\mathrm{d}I}{\mathrm{d}x}z \\ &= Vyz \\ &= V\gamma^2 \end{aligned} \tag{8}$$

To solve equations (7) and (8), which will have nearly identical solutions, we bring the expressions into the Laplace domain, where $I(s)$ and $V(s)$ are voltage and current functions in the Laplace domain.

$$\mathcal{L}\left\{\frac{\mathrm{d}^2 I}{\mathrm{d}x^2} = I\gamma^2\right\} \Rightarrow s^2 I(s) - \left(sI(x) + \frac{\mathrm{d}I(x)}{\mathrm{d}x}\right)|_{x=0} = I(s)\gamma^2$$

$$\mathcal{L}\left\{\frac{\mathrm{d}^2 V}{\mathrm{d}x^2} = V\gamma^2\right\} \Rightarrow s^2 V(s) - \left(sV(x) + \frac{\mathrm{d}V(x)}{\mathrm{d}x}\right)|_{x=0} = V(s)\gamma^2$$

We may solve these expressions for $I(s)$ and $V(s)$ respectively.

$$\begin{aligned} I(s) &= \frac{\left(sI(x) + \frac{\mathrm{d}I(x)}{\mathrm{d}x}\right)|_{x=0}}{s^2 - \gamma^2} \\ &= I(x)|_{x=0}\left[\frac{s}{s^2-\gamma^2}\right] + \frac{\mathrm{d}I(x)}{\mathrm{d}x}|_{x=0}\left[\frac{1}{s^2-\gamma^2}\right] \\ V(s) &= \frac{\left(sV(x) + \frac{\mathrm{d}V(x)}{\mathrm{d}x}\right)|_{x=0}}{s^2 - \gamma^2} \\ &= V(x)|_{x=0}\left[\frac{s}{s^2-\gamma^2}\right] + \frac{\mathrm{d}V(x)}{\mathrm{d}x}|_{x=0}\left[\frac{1}{s^2-\gamma^2}\right] \end{aligned}$$

We now recognize two important Laplace transform properties: sinh and cosh. We break both of these functions down into they exponential constituents and show their Laplace transforms.

$$\sinh(\gamma x) = \frac{e^{\gamma x} - e^{-\gamma x}}{2}$$

The Laplace transform can be computed and manipulated.

$$\begin{aligned} \mathcal{L}\left\{\frac{e^{\gamma x} - e^{-\gamma x}}{2}\right\} &= \frac{1}{2}\left[\frac{1}{s-\gamma} - \frac{1}{s+\gamma}\right] \\ &= \frac{1}{2}\left[\frac{s+\gamma}{(s-\gamma)(s+\gamma)} - \frac{s-\gamma}{(s+\gamma)(s-\gamma)}\right] \\ &= \frac{\gamma}{s^2-\gamma^2} \end{aligned}$$

We show a similar property for $\cosh(\gamma x)$.

$$\cosh(\gamma x) = \frac{e^{\gamma x} + e^{-\gamma x}}{2}$$

The Laplace transform can be computed and manipulated.

$$\begin{aligned}\mathcal{L}\left\{\frac{e^{\gamma x}+e^{-\gamma x}}{2}\right\} &= \frac{1}{2}\left[\frac{1}{s-\gamma}+\frac{1}{s+\gamma}\right] \\ &= \frac{1}{2}\left[\frac{s+\gamma}{(s-\gamma)(s+\gamma)}+\frac{s-\gamma}{(s+\gamma)(s-\gamma)}\right] \\ &= \frac{s}{s^2-\gamma^2}\end{aligned}$$

We use these properties to solve for $I(x)$ and $V(x)$. For $I(x)$, we have the following.

$$\begin{aligned}I(x) &= \mathcal{L}^{-1}\left\{I(x)|_{x=0}\left[\frac{s}{s^2-\gamma^2}\right]+\frac{\mathrm{d}I(x)}{\mathrm{d}x}|_{x=0}\left[\frac{1}{s^2-\gamma^2}\right]\right\} \\ &= \mathcal{L}^{-1}\left\{I(x)|_{x=0}\left[\frac{s}{s^2-\gamma^2}\right]+\frac{\mathrm{d}I(x)}{\mathrm{d}x}|_{x=0}\frac{1}{\gamma}\left[\frac{\gamma}{s^2-\gamma^2}\right]\right\} \\ &= [I(x)|_{x=0}]\cosh(\gamma x)+\left[\frac{\mathrm{d}I(x)}{\mathrm{d}x}|_{x=0}\frac{1}{\gamma}\right]\sinh(\gamma x)\end{aligned}$$

And for $V(x)$, we have the following.

$$\begin{aligned}V(x) &= \mathcal{L}^{-1}\left\{V(x)|_{x=0}\left[\frac{s}{s^2-\gamma^2}\right]+\frac{\mathrm{d}V(x)}{\mathrm{d}x}|_{x=0}\left[\frac{1}{s^2-\gamma^2}\right]\right\} \\ &= \mathcal{L}^{-1}\left\{V(x)|_{x=0}\left[\frac{s}{s^2-\gamma^2}\right]+\frac{\mathrm{d}V(x)}{\mathrm{d}x}|_{x=0}\frac{1}{\gamma}\left[\frac{\gamma}{s^2-\gamma^2}\right]\right\} \\ &= [V(x)|_{x=0}]\cosh(\gamma x)+\left[\frac{\mathrm{d}V(x)}{\mathrm{d}x}|_{x=0}\frac{1}{\gamma}\right]\sinh(\gamma x)\end{aligned}$$

For each function, we substitute in the derivatives from equations (2) and (4).

$$\begin{aligned}I(x) &= [I(0)]\cosh(\gamma x)+\left[V(0)\frac{y}{\gamma}\right]\sinh(\gamma x) \\ V(x) &= [V(0)]\cosh(\gamma x)+\left[\frac{z}{\gamma}I(0)\right]\sinh(\gamma x)\end{aligned}$$

Next we define $Z_c=\sqrt{\frac{z}{y}}$ (Ω) as the characteristic impedance of the line, where $\frac{y}{\gamma}=\frac{1}{Z_c}$ and $\frac{z}{\gamma}=Z_c$. We also use Figure 1 to define $I(0)=I_2$ and $V(0)=V_2$.

$$\begin{aligned}I(x) &= I_2\cosh(\gamma x)+\frac{V_2}{Z_c}\sinh(\gamma x) \\ V(x) &= V_2\cosh(\gamma x)+Z_cI_2\sinh(\gamma x)\end{aligned}$$

We may write these equations in terms for receiving end and sending end variables. And we set $x=l$, which corresponds to the transmission line length.

$$\begin{aligned}\begin{bmatrix}V_s\\ I_s\end{bmatrix} &= \begin{bmatrix}\cosh(\gamma l) & Z_c\sinh(\gamma l)\\ \frac{\sinh(\gamma l)}{Z_c} & \cosh(\gamma l)\end{bmatrix}\begin{bmatrix}V_r\\ I_r\end{bmatrix} \\ &= \begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}V_r\\ I_r\end{bmatrix}\end{aligned} \quad (9)$$

We now essentially have a two port model, and we can convert the distributed parameter system into a π-equivalent model with *lumped* parameters. In this model, $Z_L=R+jX$ is the line impedance, placed in the center of the line, and $\frac{Y}{2}=\frac{jB}{2}$ is the shunt capacitance. Half of the total capacitance is lumped at the beginning of the line while half of the capacitance is lumped at the end of the line.

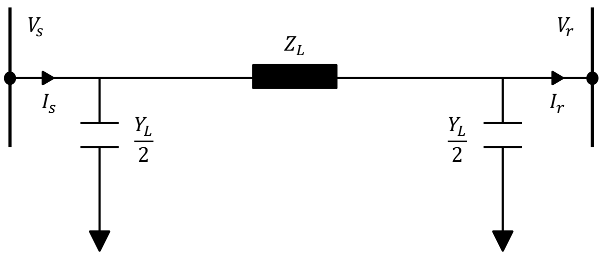

Figure 2: Single-phase π-model transmission line equivalent circuit

The two port model for this system is relatively easy to construct. We write the voltage at the sending bus in terms of the voltage at the receiving bus minus the voltage drop across the series impedance.

$$V_s = V_r + \left(I_r + V_r \frac{Y_L}{2}\right) Z_L$$

Next, we write the sending end current as the sum of the receiving end current minus the shunt losses.

$$\begin{aligned} I_s &= I_r + V_r \frac{Y_L}{2} + V_s \frac{Y_L}{2} \\ &= I_r + V_r \frac{Y_L}{2} + \left(V_r + \left(I_r + V_r \frac{Y_L}{2}\right) Z_L\right) \frac{Y_L}{2} \end{aligned}$$

We may now rearrange these expressions to build the two-port model.

$$\begin{aligned} V_s &= V_r \left[1 + \frac{Z_L Y_L}{2}\right] + I_r \left[Z_L\right] \\ I_s &= V_r \left[Y_L + \frac{Z_L Y_L^2}{4}\right] + I_r \left[1 + \frac{Z_L Y_L}{2}\right] \end{aligned}$$

In matrix form, we have the following expression.

$$\begin{bmatrix} V_s \\ I_s \end{bmatrix} = \begin{bmatrix} \left[1 + \frac{Z_L Y_L}{2}\right] & \left[Z_L\right] \\ \left[Y_L + \frac{Z_L Y_L^2}{4}\right] & \left[1 + \frac{Z_L Y_L}{2}\right] \end{bmatrix} \begin{bmatrix} V_r \\ I_r \end{bmatrix} \tag{10}$$

We may now relate parameters between the lumped model and distributed model directly. We consider the upper right entries of the matrices in (9) and (10).

$$\begin{aligned} Z_L &= Z_c \sinh\left(\gamma l\right) \\ &= \frac{\gamma l Z_c \sinh\left(\gamma l\right)}{\gamma l} \\ &= z l \frac{\sinh\left(\gamma l\right)}{\gamma l} \end{aligned}$$

Next we consider the lower right entries of these matrices.

$$\begin{aligned} 1 + \frac{Z_L Y_L}{2} &= \cosh\left(\gamma l\right) \\ 1 + z l \frac{\sinh\left(\gamma l\right)}{\gamma l} \frac{Y_L}{2} &= \cosh\left(\gamma l\right) \\ Y_L &= \frac{2\gamma l}{z l} \frac{\cosh\left(\gamma l\right) - 1}{\sinh\left(\gamma l\right)} \\ &= y l \frac{\tanh\left(\frac{\gamma l}{2}\right)}{\frac{\gamma l}{2}} \end{aligned}$$

We therefore use the following expressions to convert from a distributed model to a lumped model.

$$Z_L = zl\frac{\sinh(\gamma l)}{\gamma l} \tag{11}$$

$$Y_L = yl\frac{\tanh\left(\frac{\gamma l}{2}\right)}{\frac{\gamma l}{2}} \tag{12}$$

We note that if $\frac{\gamma l}{2}$ is small, we may approximate the lumped parameters by employing the small angle assumption of $\sinh(\gamma l) \approx \gamma l$ and $\tanh\left(\frac{\gamma l}{2}\right) \approx \frac{\gamma l}{2}$. Therefore, the lumped line series impedance and shunt susceptance is simply the product of the characteristic impedance or susceptance times the line length, respectively.

$$Z_L = zl \tag{13}$$

$$Y_L = yl \tag{14}$$