A Passivity Enforcement Technique for Forced Oscillation Source Location

Samuel Chevalier, *Student Member, IEEE,* Petr Vorobev, *Member, IEEE,* Konstantin Turitsyn, *Member, IEEE*

***Abstract*—This paper develops a systematic framework for analyzing how low frequency forced oscillations propagate in electric power systems. Using this framework, the paper shows how to mathematically justify the so-called Dissipating Energy Flow (DEF) forced oscillation source-location technique, and the DEF's specific deficiencies are pinpointed. Leveraging incremental passivity enforcement, a set of simple inference problems are introduced whose solutions can be used in conjunction with the proposed propagation framework in order to enhance the effectiveness of the DEF method. The proposed techniques are illustrated on the IEEE 39-bus New England test system.**

***Index Terms*—Forced oscillations, inverse problems, passivity, phasor measurement unit (PMU), power system dynamics.**

I. Introduction

FORCED oscillations (FOs) are still a problematic reality in modern electric power systems. Caused by extraneous periodic perturbations, FOs can compromise system security, degrade system performance, and resonantly excite poorly damped interarea modes [1]–[3]. Despite widescale deployment of Phasor Measurement Units (PMUs) across the US high voltage transmission network, locating the sources of these FOs remains a challenging task due to their sporadic nature, speed of propagation, and inability to be predicted by the system operators' dynamical models. Since the most effective way for dealing with a FO is to locate the source and disconnect it from service [4], an effective source-location technique is an indespensible smart grid application.

Of the many source-location techniques currently available in the academic marketplace [5], the so-called Dissipating Energy Flow (DEF) method has enjoyed some of the most successful testing results, both in simulation environments [4] and in real-time applications [6]. The method was originally developed by Chen et al. [7], but its underlying mathematics leverage the Lyapunov functions from [8]. Despite its success, when its underlying modeling assumptions are violated, the method may perform poorly [9]–[11]. Additionally, the method has lacked a generalized framework which is capable of providing a system-wide justification for its usage.

Despite its inadequacies, the DEF's excellent performance in real-time application at ISO New England strongly implies that further research should be performed in order to more systematically characterize the method. Shortcomings of the DEF method have been analyzed in [9], and [11] has recommended using passivity theory to interpret the method from a new mathematical perspective, but no recommendations have been made in terms of how to enhance the method's performance (aside from practical implementation suggestions in [4]). To make such recommendations, a systematic framework is needed in order to thoroughly study the DEF. Accordingly, the primary contributions of this paper are as follows:

1) Using the Frequency Response Function (FRF) analysis proposed in [10], we leverage a variety of tools from AC circuit theory in order to develop a linearized framework for analyzing oscillation propagation at the system level.
2) We subsequently use the proposed framework, along with the passivity observations from [11], to theoretically justify the DEF method, predict when it will fail, and show that there exists no other quadratic passivity transformation which will render all components of the entire network passive in a classical power system.
3) A set of inference problems, some with DEF-based passivity constraints, is used to construct the proposed propagation model. Using this model, a simple optimization problem is solved to locate the source of a FO.

II. Perturbative Network Model Derivation

In this section, we introduce a linearized network model which is particularly useful for analyzing FO propagation in power systems. We refer to it as a "perturbative" model since it captures the network's response to small perturbations.

A. Complex Admittance Matrices

We consider a power system component, shown in Fig. 1, whose dynamics are governed according to the DAE set

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{u}_v) \tag{1a}$$
$$\mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u}, \mathbf{u}_v), \tag{1b}$$

where inputs $\mathbf{u}_v = [V_r,\, V_i]$ and outputs $\mathbf{y} = [I_r,\, I_i]$ are vectors of real and imaginary voltages and currents, respectively.

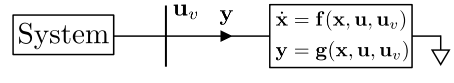

Fig. 1. DAE modeled component tied to a larger power system.

We linearize this model around a steady state operating point to linearly relate the voltage and current perturbations:

$$\Delta\dot{\mathbf{x}} = A\Delta\mathbf{x} + B\Delta\mathbf{u}_v \tag{2a}$$
$$\Delta\mathbf{y} = C\Delta\mathbf{x} + D\Delta\mathbf{u}_v. \tag{2b}$$

S. Chevalier is with Department of Mechanical Engineering, Massachusetts Institute of Technology. E-mail: schev@mit.edu

P. Vorobev is with the Skolkovo Institute of Science and Technology (Skoltech), Moscow, Russia. E-mail: p.vorobev@skoltech.ru

K. Turitsyn was with Department of Mechanical Engineering, Massachusetts Institute of Technology. E-mail: turitsyn@gmail.com

Assuming (2) is BIBO stable, its Fourier transform admits the Frequency Response Function (FRF) of the component [10]:

$$\tilde{\mathbf{y}} = \underbrace{\left(C(j\Omega\mathbb{1} - A)^{-1}B + D\right)}_{Y(j\Omega)} \tilde{\mathbf{u}}_v, \tag{3}$$

where Ω is the angular frequency of the input and output signals, and $Y \triangleq Y(j\Omega) \in \mathbb{C}^{2\times 2}$ is referred to as the admittance matrix relating voltage $\tilde{\mathbf{u}}_v \in \mathbb{C}^{2\times 1}$ and current $\tilde{\mathbf{y}} \in \mathbb{C}^{2\times 1}$ perturbations. These perturbations can be given in rectangular $(\tilde{V}_r, \tilde{V}_i)$ or polar $(\tilde{\mathrm{V}}, \tilde{\theta})$ coordinates depending on convenience. In this paper, we will primarily consider the effects of the FRF matrix at the relevant forcing frequency Ω_d of the FO.

B. Network Modeling

Consider a power system network whose graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ has edge set $\mathcal{E}$, $|\mathcal{E}| = m$, vertex set $\mathcal{V}$, $|\mathcal{V}| = n$, and directed nodal incidence matrix $E \in \mathbb{R}^{m\times n}$. We build an augmented incidence matrix $E_a \in \mathbb{R}^{2m\times 2n}$. This matrix is constructed by taking E and replacing all values of 1 with the 2×2 identity matrix $\mathbb{1}_2$ and all values of 0 with a 2×2 zero matrix $\mathbf{0}$:

$$E = \begin{bmatrix} 1 & -1 & \\ 0 & 1 & \\ \vdots & & \ddots \end{bmatrix}, \quad E_a = \begin{bmatrix} \mathbb{1}_2 & -\mathbb{1}_2 & \\ \mathbf{0} & \mathbb{1}_2 & \\ \vdots & & \ddots \end{bmatrix}. \tag{4}$$

Considering voltage and current perturbation phasors such as $\tilde{\mathbf{u}}_v$ and $\tilde{\mathbf{y}}$ from (3), we define the vector $\mathbf{V}_b \in \mathbb{C}^{2n\times 1}$ as the vector of rectangular bus voltage perturbation phasors, and we define the vector $\mathbf{I}_l \in \mathbb{C}^{2m\times 1}$ as the vector of rectangular line current perturbation phasors, where the convention of the positive line current flows agrees with the direction of the augmented incidence matrix:

$$\mathbf{V}_b = \begin{bmatrix} \tilde{V}_{r,1} \\ \tilde{V}_{i,1} \\ \vdots \\ \tilde{V}_{r,n} \\ \tilde{V}_{i,n} \end{bmatrix}, \tag{5} \qquad \mathbf{I}_l = \begin{bmatrix} \tilde{I}_{r,1} \\ \tilde{I}_{i,1} \\ \vdots \\ \tilde{I}_{r,m} \\ \tilde{I}_{i,m} \end{bmatrix}. \tag{6}$$

Admittance matrices $Y_{l,1}, ..., Y_{l,m}$, $Y_{l,i} \in \mathbb{R}^{2\times 2}$, associated with the m transmission lines, are placed diagonally in the matrix $Y_L \in \mathbb{R}^{2m\times 2m}$ such that

$$Y_L = \begin{bmatrix} Y_{l,1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & Y_{l,m} \end{bmatrix}. \tag{7}$$

Line current and bus voltage perturbations obey Ohm's law:

$$Y_L E_a \mathbf{V}_b = \mathbf{I}_l. \tag{8}$$

Admittance matrices $Y_{s,1}, ..., Y_{s,n}$, $Y_{s,i} \in \mathbb{C}^{2\times 2}$, associated with shunt elements at each of the n buses, are placed diagonally in the matrix $Y_S \in \mathbb{C}^{2n\times 2n}$, such that

$$Y_S = \begin{bmatrix} Y_{s,1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & Y_{s,n} \end{bmatrix}. \tag{9}$$

These shunt admittances are not simply capacitors or inductors; they can represent generators, loads, or any other terminal element in the system and can be constructed via (3). If multiple elements are connected in parallel, such as a generator and its station load, their admittances can be modeled independently and summed to compute the aggregate shunt admittance. The shunt matrix Y_S may be used to compute the shunt current injections[1] $\mathbf{I}_s \in \mathbb{C}^{2n\times 1}$ via

$$\mathbf{I}_s = Y_S \mathbf{V}_b. \tag{10}$$

As outlined in [10], when analyzing a network with this representation, FOs show up like current sources at their respective source buses. For a system experiencing a single FO, there will be a single current source driving the entire network. We define sparse FO vector of current injections $\mathbf{J} \in \mathbb{C}^{2n\times 1}$. If bus k contains the sole source of a FO, then

$$\mathbf{J}_i = 0, \ \forall i \notin \{2k-1, 2k\}. \tag{11}$$

Source injections in $\mathbf{J}$ obey the same current convention as $\mathbf{I}_s$. The network obeys KCL, i.e., all nodal currents sum to 0:

$$\mathbf{J} + \mathbf{I}_s + E_a^{\dagger}\mathbf{I}_l = \mathbf{0}. \tag{12}$$

We define $\mathbf{I}_I \triangleq -E_a^{\dagger}\mathbf{I}_l$ to be the aggregate current injection at each node: it represents the sum of the source current injection at each bus plus the shunt current flowing to ground. Via conservation of current at each bus, we have

$$\begin{aligned} \mathbf{J} &= \mathbf{I}_I + \mathbf{I}_s && (13a) \\ &= -E_a^{\dagger}\mathbf{I}_l + Y_S \mathbf{V}_b && (13b) \\ &= \left(-E_a^{\dagger} Y_L E_a + Y_S\right) \mathbf{V}_b. && (13c) \end{aligned}$$

The block-Hermitian dynamic nodal admittance (or augmented dynamic Y-bus) matrix $Y_B \in \mathbb{C}^{2n\times 2n}$ is this defined to be

$$Y_B = -E_a^{\dagger} Y_L E_a + Y_S. \tag{14}$$

Assuming there is a single FO in the system, it is instructive to rewrite (13c) with partitioned matrices and vectors, where the system has been renumbered such that the source bus is bus 1, and the current injection has a value of $\mathcal{I} \in \mathbb{C}^{2\times 1}$:

$$\begin{aligned} \mathbf{J} &= Y_B \mathbf{V}_b && (15a) \\ \begin{bmatrix} \mathcal{I} \\ \hline \mathbf{0} \end{bmatrix} &= \left[\begin{array}{c|c} Y_{B1} & Y_{B2} \\ \hline Y_{B3} & Y_{B4} \end{array}\right] \begin{bmatrix} \mathbf{V}_s \\ \hline \mathbf{V}_{ns} \end{bmatrix}, && (15b) \end{aligned}$$

where $\mathbf{V}_s \in \mathbb{C}^{2\times 1}$ represents voltage perturbations at the source bus, $\mathbf{V}_{ns} \in \mathbb{C}^{(2n-2)\times 1}$ represents voltage perturbations at all other buses, and $\mathbf{V}_b = \mathbf{V}_s^\frown \mathbf{V}_{ns}$. While $\mathcal{I}$ represents the true source current injection, we may also define $\mathcal{I}'$ as the sum of the source current at the source bus plus its shunt injection:

$$\mathcal{I}' = \mathcal{I} + Y_{s,1}\mathbf{V}_s. \tag{16}$$

Correspondingly, we say that Y_{B1}' contains no shunt element, and $\mathcal{I}'$ is the current *directly measured at the source bus flowing into the network*; we note that it is equal to the first two elements of $\mathbf{I}_I$. We now restate (15a)-(15b) with this update:

$$\begin{aligned} \mathbf{J}' &= Y_B' \mathbf{V}_b && (17a) \\ \begin{bmatrix} \mathcal{I}' \\ \hline \mathbf{0} \end{bmatrix} &= \left[\begin{array}{c|c} Y_{B1}' & Y_{B2} \\ \hline Y_{B3} & Y_{B4} \end{array}\right] \begin{bmatrix} \mathbf{V}_s \\ \hline \mathbf{V}_{ns} \end{bmatrix}. && (17b) \end{aligned}$$

[1] Shunt currents flowing out of the circuit to ground are defined as positive.

A simple Kron reduction can be performed in order to determine the effective admittance "seen" by the current source:

$$I^{0} = \underbrace{\left(Y^{0}_{B1} \quad Y_{B2} Y_{B4}^{1} Y_{B3}\right)}_{Y_N} V_s; \tag{18}$$

where $Y_N \; 2 \; C^{2 \; 2}$ is a 2 2 complex aggregate network admittance matrix and V_s is the resulting voltage caused by the current injection I^0 interacting with the aggregate network dynamics codified in Y_N. In this model, since voltage perturbations are considered a response to rouge current injections, it is helpful to rewrite (18) with voltage as a function of current:

$$V_s = \quad Z_N I^0; \tag{19}$$

where $Z_N = Y_N^{1}$ is the aggregate network impedance. In other words, the current injection I^0 gives rise to the network voltages, and the vector V_b in (17a) is not arbitrary: the Kron reduction of (18) is only meaningful when V_b acts as a solution to the linear system $J^0 = Y^0_B V_b$, i.e. $V_b = \quad (Y^0_B)^{1} J^0$.

Definition 1. We refer to the admittance matrix Y_N of (18) as the system's dynamic Ward equivalent (DWE) admittance.

C. Quadratic Energy Considerations

As with any network which obeys Kirchhoff's laws, Tellegen's theorem is also obeyed: the sum of the products of branch (including shunt branches) potential differences and branch flows is equal to 0. Accordingly,

$$0 = (E_a V_b)^y I_l + V_b^y (I_s + J) \tag{20a}$$

$$= V_b^y \; E_a^y I_l + I_s + J \tag{20b}$$

$$= V_b^y (Y_B \quad Y_B) V_b; \tag{20c}$$

where (20a) is the statement of Tellegen's theorem, (20b) is the conservation of current, and (20c) is the resulting proof. As a consequence of this theorem, there exist a family of quadratic functionals for which conservation laws can be formulated. An obvious one is the "real power," $Ref VI^y g$, that is consumed only on the elements with positive resistance. In the context of FOs, an alternative interpretation of the conservation of power can be acquired by manipulating (17b) in order to define another (arbitrary) type of quadratic power[2]. The key observation is that this new quadratic power will be conserved throughout the network. To show why, we consider matrix $Q \; 2 \; C^{2n \; 2n}$ with block diagonal sub-matrices $Q_b \; 2 \; C^{2 \; 2}$:

$$Q = \begin{bmatrix} Q_b & & 0 \\ & \ddots & \\ 0 & & Q_b \end{bmatrix}: \tag{21}$$

We now left multiply (17a) by $V_b^y Q$, which represents the application of a quadratic energy function:

$$V_b^y Q J^0 = V_b^y Q Y^0_B V_b \tag{22a}$$

$$V_s^y Q_b I^0 = V_b^y \; Q E_a^y Y_L E_a \; V_b + V_b^y (Q Y^0_S) V_b: \tag{22b}$$

[2]The term "quadratic power" is used since we are multiplying voltages and currents, but the quadratic quantity doesn't necessarily have the interpretation of physical power. It can also be interpreted as an "energy function". Quadratic energy and quadratic power are therefore used interchangeably.

Different choices for matrix Q_b correspond to different energy function applications, but in each case, the quadratic quantity is conserved. For example, if Q_b is chosen such that

$$Q_b = \begin{bmatrix} 0 & \frac{1}{j} \\ \frac{1}{j} & 0 \end{bmatrix}; \tag{23}$$

then the associated energy function corresponds to the DEF method [11]. Under DEF assumptions, lines and loads are rendered lossless, i.e., $(Q_b Y) + (Q_b Y)^y = 0$. Thus, in taking the real part, (22b) simplifies to

$$\underbrace{Ref V_s^y Q_b I^0 g}_{\text{Source Energy}} + \sum_{i=2}^{n} \underbrace{Ref V_{ns;i}^y \; (Q_b Y_{s;i}) V_{ns;i} g}_{\text{Generator Damping Contributions}} = 0: \tag{24}$$

where V_s and $V_{ns;i}$ are the source and i^{th} non-source bus voltage perturbation vectors, respectively. The formulation of (24) further clarifies the DEF's functionality: since all the damping energy consumed by generators is positive [11], the source energy is necessarily negative and can be traced back to the single, negative source. The DEF technique, therefore, is based on tracking a particular type of quadratic power in the network. When constructing the system's quadratic energy function, we are not restricted to choosing just a Q matrix. We may also introduce matrix P whose structure matches Q. For example, we may set $U_b = P^{1} V_b$, left multiply (17a) by $U_b^y Q$, and insert a $P P^{1}$ term. Updating (24) yields:

$$\underbrace{Ref U_s^y Q_b I^0 g}_{\text{Source Energy}} + \sum_{i=2}^{n} \underbrace{Ref U_{ns;i}^y \; (Q_b Y_{s;i} P_b) U_{ns;i} g}_{\text{Generator Damping Contributions}} = 0: \tag{25}$$

III. Energy Function Analysis

The DEF method can be interpreted as the application of a specific quadratic energy function to all elements of a network. Reference [11], which reviews relevant passivity concepts, explains how this quadratic energy function can be interpreted as a matrix transformation (called a "passivity transformation") which attempts to render system elements incrementally passive. Generally, a passive component is one which can only dissipate and store, but not produce, physical power. To avoid confusion, we point out that this paper discusses the so-called incremental passivity, which refers to the passive nature of a system's incremental change from its equilibrium. This is also known as "shifted passivity". We stress that physical passivity of an element does not imply incremental passivity of the transformed system. In the remainder of the paper, the term passivity always refers to incremental passivity.

Reference [11] shows that the DEF energy function is inadequate for lossy network elements. While it may be tempting to develop a new passivity transformation which is suitable for lossy networks, in this section, we use passivity theory to prove that no constant quadratic energy function exists for the classical model of a multimachine power system.

A. Basis Matrices

To aide in the energy function analysis and inference techniques, we define a useful set of basis matrices.

Definition 2. We define orthogonal ($A^{-1} = A^\dagger$) basis matrices

$$\overline{T}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \overline{T}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix};$$

$$\overline{T}_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \overline{T}_4 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix};$$

and the set $T = \{\overline{T}_1;\overline{T}_2;\overline{T}_3;\overline{T}_4\}$. Set T spans region $\mathbb{R}^{2\times 2}$.

Lemma 1. There exists no non-singular matrix $\in \mathbb{C}^{2\times 2}$ for which, simultaneously,

$$\mathrm{Re}\{v^\dagger \overline{T}_i v\} \geq 0;\ \forall v \in \mathbb{C}^{2\times 1};\ \forall i \in \{1\} \quad (26)$$

$$\mathrm{Re}\{v^\dagger \overline{T}_i v\} = 0;\ \forall v \in \mathbb{C}^{2\times 1};\ \forall i \in \{2;3;4\}: \quad (27)$$

Proof. We write as the sum of its diagonal ($_d$) and off-diagonal ($_o$) component matrices: $= {}_d + {}_o$. Since $\mathrm{Re}\{v^\dagger(\overline{T}_i)v\} = 0;\ \forall v \in \mathbb{C}^{2\times 2}$ is equivalent to stating that $\overline{T}_i + (\overline{T}_i)^\dagger = 0$, the constraints on caused by $\overline{T}_2$, $\overline{T}_3$, and $\overline{T}_4$ from (27) may be stated as

$$\overline{T}_2 \to \quad {}_d = -{}^\dagger_d; \quad {}_o = {}^\dagger_o; \quad (28)$$

$$\overline{T}_3 \to \quad K\,{}_d = -{}^\dagger_d K; \quad K\,{}_o = -{}^\dagger_o K; \quad (29)$$

$$\overline{T}_4 \to \quad K\,{}_d = {}^\dagger_d K; \quad K\,{}_o = -{}^\dagger_o K; \quad (30)$$

where $K \equiv \overline{T}_3$ is defined to be the reversal matrix in [12]. Accordingly, $_d$ must be simultaneously skew-Hermitian, skew-perhermitian and perhermitian, respectively [12]. Necessarily, $_d = 0$. The matrix $_o$ must be simultaneously Hermitian and skew-perhermitian. Necessarily, $j\overline{T}_4;\ \in \mathbb{R}^1$, is the only matrix which fits this description. We define

$$^\star = j\overline{T}_4 \quad (31)$$

as the only matrix which uniformly satisfies (27). We apply $= {}^\star$ to (26) and consider the eigenvalues of the matrix $\overline{T}_1{}^\star + (\overline{T}_1{}^\star)^\dagger = 2j\overline{T}_4$:

$$\det\left(2j\overline{T}_4 - \mathrm{diag}\{\ \}\right) = {}^2 - 4{}^2; \quad (32)$$

$$= \pm 2 \quad (33)$$

which violates (26). Since $\overline{T}_1{}^\star + (\overline{T}_1{}^\star)^\dagger$ is an indefinite matrix but * is the only matrix which satisfies (27), the theorem has been proved. □

Corollary 1. * is a solution to $P{}^\star + (P{}^\star)^\dagger = 0$ for any matrix P which may be written as $P = \sum_{i=2}^{4} {}_i\overline{T}_i;\ {}_i \in \mathbb{R}^1$.

Corollary 2. The results of Lemma 1 stand if $\overline{T}_i$ is right multiplied by transformation matrix M instead of left multiplied by transformation matrix . There exists no non-singular matrix $M \in \mathbb{C}^{2\times 2}$ for which the following simultaneously hold:

$$\mathrm{Re}\{v^\dagger M\overline{T}_i v\} \geq 0;\ \forall v \in \mathbb{C}^{2\times 2};\ \forall i \in \{1\} \quad (34)$$

$$\mathrm{Re}\{v^\dagger M\overline{T}_i v\} = 0;\ \forall v \in \mathbb{C}^{2\times 2};\ \forall i \in \{2;3;4\} \quad (35)$$

Corollary 3. By employing both non-singular matrices $\in \mathbb{C}^{2\times 2}$ and $M \in \mathbb{C}^{2\times 2}$, the solution to

$$\mathrm{Re}\{v^\dagger M\overline{T}_i v\} = 0;\ \forall v \in \mathbb{C}^{2\times 1};\ \forall i \in \{2;3;4\} \quad (36)$$

must take the form $= (j\overline{T}_4)M^\dagger$ for any $M \in \mathbb{C}^{2\times 2}$. This may be seen via the following manipulation:

$$0 = M\overline{T}_i + (M\overline{T}_i)^\dagger; \qquad \forall i \in \{2;3;4\} \quad (37a)$$

$$= \overline{T}_i(M^{\dagger\,-1}) + (M^{-1})^\dagger\overline{T}_i^\dagger; \qquad \forall i \in \{2;3;4\} \quad (37b)$$

Since (37b) may only be solved by $M^{\dagger\,-1} = j\overline{T}_4$, per Lemma 1, we have that $= (j\overline{T}_4)M^\dagger$ must be satisfied.

B. Quadratic Energy Functions in a Classical Power System

We now assume the classical model of a lossy multimachine power system model [13] and allow for constant power loads to be present. The forms of the FRFs associated with constant power loads (Y_p), constant impedance lines/shunts (Y_z), and classical generators (Y_g) are given in [11]. The set of plausible FRFs associated with these three elements may be constructed according to the following basis matrix combinations:

$$Y_p = \sum_{i=2;3} a_i\overline{T}_i;\ a_i \in \mathbb{R}^1 \quad (38a)$$

$$Y_z = \sum_{i=1;4} a_i\overline{T}_i;\ a_i \in \mathbb{R}^1 \quad (38b)$$

$$Y_g = \sum_{i=2;3;4} (a_i + jb_i)\overline{T}_i;\ a_i;b_i \in \mathbb{R}^1: \quad (38c)$$

The FRF of a classical generator is derived and fully explained in [10], and, for convenience, is explicitly re-stated here:

$$Y_g = (\ )\underbrace{\begin{bmatrix} \sin(\ )\cos(\ ) & \cos^2(\ ) \\ \sin^2(\ ) & \sin(\ )\cos(\ ) \end{bmatrix}}_{T} + \underbrace{\begin{bmatrix} 0 & \frac{1}{X_d^0} \\ -\frac{1}{X_d^0} & 0 \end{bmatrix}}_{T_X} \quad (39)$$

$$(\ ) = \frac{E^{02}}{X_d^{02}} \frac{M(j)^2 + \frac{V_t E^0}{X_d^0}\cos(\ ) - j(D)}{\left(\frac{V_t E^0}{X_d^0}\cos(\ ) - M^2\right)^2 + (D)^2}: \quad (40)$$

where is the generator's absolute rotor angle and $(\ ) \in \mathbb{C}^1$ is a complex frequency dependent parameter. Given that any FRF matrix Y may be written as the complex sum of the four basis matrices, we have the following useful definition:

Definition 3. We assume $Y = \sum_{i=1}^{4}(a_i + jb_i)\overline{T}_i$. Using matrices M and from Corollary 3, where $= (j\overline{T}_4)M^\dagger$, we define $P^\star = \mathrm{Re}\{u^\dagger(MY)u\}$ as the dissipating power for input vector u. We further define two other types of quadratic power: resistive power $P_r^\star$ and damping power $P_d^\star$, where $P^\star = P_r^\star + P_d^\star$, and

$$P_r^\star = \mathrm{Re}\{u^\dagger M(a_1\overline{T}_1)u\}$$

$$P_d^\star = \mathrm{Re}\{u^\dagger M(jb_2\overline{T}_2 + jb_3\overline{T}_3 + jb_4\overline{T}_4)u\}.$$

We now prove that a perturbative system model containing elements (38a)-(38c) cannot be rendered passive under any quadratic passivity transformation. In other words, there is no quadratic quantity that is dissipated by all elements present in common networks.

Theorem 1. There exist no non-singular matrices $M \in \mathbb{C}^{2\times 2}$ and $\in \mathbb{C}^{2\times 2}$ for which

$$MY + (MY)^\dagger \succeq 0;\quad \forall Y \in \{Y_p; Y_z; Y_g\}: \quad (41)$$

Proof. The FRF of a strictly reactive element, such as matrix T_X in (39), is $\propto \overline{T}_4$ while the FRF of a strictly capacitive element is $\propto \overline{T}_4$. The only way for $M\overline{T}_4 + (M\overline{T}_4)^\dagger \succeq 0$ and $-M\overline{T}_4 - (M\overline{T}_4)^\dagger \succeq 0$ to be simultaneously true is for $M\overline{T}_4 + (M\overline{T}_4)^\dagger \equiv 0$. Since both reactive and capacitive elements appear in classical power systems, $\overline{T}_4$ must be lossless under the desired passivity transformation.

We now consider some classical generator whose damping characteristics are sufficiently small ($D \approx 0$), such that is a real parameter. In this case, the matrix $MY_g + (MY_g)^{\dagger}$ reduces to $MT + (MT)^{\dagger}$ since T_X must be a lossless element according to the previous conclusion about $\overline{T}_4$.

We define the squared electromechanical resonant frequency associated with the classical generator as $= \frac{VE^0}{MX_d^0}\cos(\,)$. For some, $(\,_r) = (\,_r + \,)$. We must therefore ensure that $MT + (MT)^{\dagger} \geq 0$ and $MT - (MT)^{\dagger} \geq 0$, respectively, when ensuring the generator's passivity on either side of the resonant peak. The only way for these statements to be simultaneously true is for $MT + (MT)^{\dagger} = 0$. To accomplish this, we consider the numerical structure of T for two plausible rotor angle values: $_1 = 0$ and $_2 = \frac{}{4}$:

$$T(\,_1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}\left(\overline{T}_4 - \overline{T}_3\right); \qquad (42)$$

$$T(\,_2) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{4}\left(\overline{T}_2 + \overline{T}_4\right): \qquad (43)$$

Since $M\overline{T}_4 + \left(M\overline{T}_4\right)^{\dagger} = 0$, then we must also require $M\overline{T}_3 + \left(M\overline{T}_3\right)^{\dagger} = 0$ from (42) and $M\overline{T}_2 + \left(M\overline{T}_2\right)^{\dagger} = 0$ from (43) in order to ensure that $MT + (MT)^{\dagger} = 0$. We are thus requiring that $\mathrm{Re}\{v^{\dagger} M\overline{T}_i v\} = 0;\ \forall v \in \mathbb{C}^{2\times 1};\ \forall i \in \{2;3;4\}$. As stated in Corollary 3, the only way to achieve losslessness for basis matrices 2, 3 and 4 is for $= (j\overline{T}_4)M^{\dagger}$. By employing this transformation, Lemma 1 proves that the quadratic energy associated with any element containing $\overline{T}_1$ will be rendered indefinite in sign. Since (38b) contains $\overline{T}_1$ when resistance is present in the network, then there exists no nonsingular matrices M and for which $MY + (MY)^{\dagger} \geq 0;\ \forall Y \in \{Y_p;\ Y_z;\ Y_g\}$. □

Corollary 4. By choosing $M = \overline{T}_1$, $= \,^{-1}$, and $= (j\overline{T}_4)M^{\dagger}$, we arrive at the passivity transformation implicitly employed by the DEF, as given by [11, eq. (38a,b)].

Corollary 5. The passivity transformation $= (j\overline{T}_4)M^{\dagger}$ renders constant power and reactive (capacitive or inductive) elements of the system lossless, classical generators passive (see [11, eq. (49)]), and resistive elements non-passive. Without resistive elements, the DEF method is a fully reliable source-location technique in classical power systems.

Corollary 6. Since the linearized admittance matrix Y associated with any ZIP load is purely real, i.e. $Y = \mathrm{Re}\{Y\}$, the eigenvalues of its transformed Hermitian part will be equal and opposite in value: $(MY + (MY)^{\dagger}) = \,$.

If a network has no resistive elements and is truly passive, no quadratic energy production can occur on regular network elements, so injections of energy have to be related to external sources, like FOs. This is why finding a passive energy-like form is important, and why the almost-passive form used by DEF has had so much success. To further show why the results of Theorem 1 are problematic for the DEF method, we consider the structure of (24): since the generator damping contributions are positive definite, the source energy is necessarily negative in a lossless power system. To contest this point in a lossy power system, we define block matrices $\underline{M} \in \mathbb{C}^{2n \times 2n}$ and $\underline{\;} \in \mathbb{C}^{2n \times 2n}$, where

$$\underline{M} = \begin{bmatrix} M & & 0 \\ & \ddots & \\ 0 & & M \end{bmatrix}; \quad \underline{\;} = \begin{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \end{bmatrix};$$

and whose block diagonal matrices are given by M and $= (j\overline{T}_4)M^{\dagger}$, respectively. We left multiply (17a) by $\underline{M}$ and insert $\underline{\;}\,\underline{\;}^{-1}$ on the RHS:

$$\underline{M}J^0 = \underline{M}\left(E_a^{\dagger} Y_L E_a + Y_S^0\right)\underline{\;}\,\underline{\;}^{-1} V_b: \qquad (44)$$

Defining the transformed voltage vectors $U_b = \underline{\;}^{-1} V_b$ and $U_s = \underline{\;}^{-1} V_s$, we left multiply (44) by $U_b^{\dagger}$ and simplify. We may group the dissipating power injections into their respective contributing groups (assuming lossless loads):

$$0 = \underbrace{U_s^{\dagger}\underline{M}I^0}_{\text{Source}} + \underbrace{U_b^{\dagger}(\underline{M}Y_S^0\underline{\;})U_b}_{\text{Generator}} + \underbrace{U_b^{\dagger}(\underline{M}E_a^{\dagger}Y_L E_a\underline{\;})U_b}_{\text{Network}}: \quad (45)$$

The FO source term can produce only negative damping energy, i.e. $\mathrm{Re}\{U_s^{\dagger}\underline{M}I^0\} \leq 0$, if the condition

$$\underline{M}\left(Y_S^0 + E_a^{\dagger}Y_L E_a\right)\underline{\;} + \left(\underline{M}\left(Y_S^0 + E_a^{\dagger}Y_L E_a\right)\underline{\;}\right)^{\dagger} \geq 0 \quad (46)$$

is met. If it is not met, indefinitely signed resistive energy can dominate damper winding energy absorption and the source term can, in fact, appear as a positively damped element. In this plausible situation, the DEF method will fail. Next, we show that the sign of the injected resistive energy can be negated if all system voltages are complex conjugated.

Theorem 2. Consider a lossy transmission network model. For any transformed voltage vector U_b which yields quadratic energy $\mathrm{Re}\{U_b^{\dagger}(\underline{M}(E_a^{\dagger}Y_L E_a)\underline{\;})U_b\} = P_r^{?}$, there exists conjugated vector U_b which yields an equal and opposite quadratic energy $\mathrm{Re}\{U_b^{\dagger}(\underline{M}(E_a^{\dagger}Y_L E_a)\underline{\;})U_b\} = -P_r^{?}$, where $\underline{M}$ and $\underline{\;}$, with submatrices M and , yield from Corollary 3.

Proof. We split the transmission line matrix into its conductive and susceptive parts: $Y_L = Y_G + Y_{Bx}$, where $Y_{G;i} = G_i\overline{T}_1$ and $Y_{Bx;i} = B_i\overline{T}_4$. We also define block matrices $\underline{M} = j\overline{T}_4$ and $= \overline{T}_1$ from $\underline{M}$ and $\underline{\;}$. Therefore, $\underline{M}\left(E_a^{\dagger}Y_L E_a\right)\underline{\;} = \underline{M}\left(E_a^{\dagger}(Y_G + Y_{Bx})E_a\right)$. The Hermitian part (termed H) is

$$2H = \underline{M}E_a^{\dagger}(Y_G + Y_{Bx})E_a + E_a^{\dagger}(Y_G - Y_{Bx})E_a\underline{M}^{\dagger} \quad (47a)$$

$$= \underline{M}\left(E_a^{\dagger}Y_G E_a\right) + \left(E_a^{\dagger}Y_G E_a\right)\underline{M} \quad (47b)$$

$$= 2\underline{M}\left(E_a^{\dagger}Y_G E_a\right) \quad (47c)$$

where the matrices of (47b) commute since the product of Hermitian matrices is also Hermitian. For input U_b, we consider the quadratic power $P_r^{?} = U_b^{\dagger}(\underline{M}E_a^{\dagger}Y_G E_a)U_b$. We define $\overline{G}_{ij} \equiv jG_{ij}\overline{T}_4$, where G_{ij} is the scalar line conductance connecting buses i and j, and $u_i \in U_b$ is the voltage element of U_b associated with bus i:

$$P_r^{?} = \sum_{i=1}^{n} u_i^{\dagger}\Big(\sum_{j \neq i}\overline{G}_{ij}\Big)u_i - \sum_{j\neq i}^{n}\sum_{i\neq j}^{n} u_i^{\dagger}\overline{T}_4\overline{G}_{ij}u_j \quad (48a)$$

$$= \sum_{i;j \in E}\left(u_i^{\dagger}\overline{G}_{ij}u_i + u_j^{\dagger}\overline{G}_{ij}u_j - u_i^{\dagger}\overline{G}_{ij}u_j - u_j^{\dagger}\overline{G}_{ij}u_i\right) \quad (48b)$$

$$= \sum_{i;j \in E}(u_i - u_j)^{\dagger}\overline{G}_{ij}\underbrace{(u_i - u_j)}_{u_{ij}}: \quad (48c)$$

The quadratic quantity $x^{y}\overline{G}_{ij}\,x =$ may be negated by conjugating the input (proof trivial): $x^{y}\overline{G}_{ij}\,x =$. By taking U_b as an input to $\mathrm{Re}f\,\underline{M}\,U_b^{y}(E_a^{y}Y_G E_a)U_b g$, we have

$$P_r^{?} = \sum_{i;j\ 2E} u_{ij}^{y}\,\overline{G}_{ij}\,u_{ij}\ : \qquad (49)$$

by (48c). This is true for *any* input pair U_b, U_b. □

We note that U_b cannot be chosen arbitrarily; it must represent a valid solution to the linear system of (17a). Since U_b is not itself a degree of freedom but rather a response to some current injection, statements about the mathematical characteristics of $M\,Y_N + (M\,Y_N)^{y}$ are difficult to prove using energy based arguments. In the proof of the following theorem, we assume the transformed voltage vector U_b is $U_b = U_s\ U_{ns}$, where $U_{ns} = Y_4^{\ 1}Y_3\ U_s$ from (17b).

Theorem 3. Consider a classical power system with ZIP loads and a lossy transmission network. The Kron reduced admittance (DWE) seen by a FO source is Y_N as in (18). The matrix $N_1 = \frac{1}{2}(M\,Y_N) + \frac{1}{2}(M\,Y_N)^{y}$, with matrices M and from Corollary 3, will have eigenvalues $_1$ and $_2$.

(a) If condition (46) is met, $_1$, $_2 > 0$.
(b) If $\mathrm{Re}f\,U_s^{y}M\,I^{0}g > 0$ is measured, i.e. the DEF method has failed, condition (46) is violated and $_1\ \_\ _2 < 0$.

Proof. By modifying (45), we have $U_s^{y}M\,I^{0} = U_b^{y}\underline{M}(Y_S^{0} + E_a^{y}Y_L E_a)\_U_b$, and with condition (46) met, $U_s^{y}M\,I^{0} > 0$. Since $I^{0} = Y_N V_s$, we have that $U_s^{y}M\,Y_N\ U_s > 0$ by substitution, implying that $f(M\,Y_N) + (M\,Y_N)^{y}g > 0;\ 8\ \ 2 f\ _1;\ _2 g$, proving proposition (a).

If $\mathrm{Re}f\,U_s^{y}M\,I^{0}g > 0$, then $U_s^{y}N_1 U_s > 0$ is directly implied. Accordingly, $_1\ \_\ _2 < 0$. By the conservation argument presented in the proof of proposition (a), if $U_s^{y}N_1 U_s > 0$, then $\mathrm{Re}f\,U_b^{y}\underline{M}(Y_S^{0} + E_a^{y}Y_L E_a)\_U_b g < 0$ implying the violation of condition (46), proving proposition (b). □

While resistive elements of power systems can drive an eigenvalue of the system's transformed Kron admittance N_1 negative, it cannot drive both eigenvalues negative. Mathematically, if $\det(N_1)\ \ 0$ then $\mathrm{trace}(N_1)\ \ 0$.

IV. A Passivity Enforced Inference Technique for Performing Source Location

The implicit goal of the DEF method is to locate the source of negative damping in the system. When resistive elements are introduced to the network, the goal becomes obfuscated because the source appears passive, and thus, positively damped. As a corollary to Theorem 3 though, the negative[3] DWE of the source cannot be *physically* passive. A key contribution of this paper recognizes that while generators continue to be physically passive, the source only appears[4] passive when the DEF method fails. The goal of this section is to determine the generator whose dynamics appear, but are not physically, passive.

Given complex input (v) and output (i) vectors for the system in (3), the admittance $Y\ 2\ C^{2\ \ 2}$ cannot be directly inferred. In order to construct this matrix, 8 coefficients (a_i; b_i; $8i\ 2 f\,1;2;3;4g$) are necessary:

$$\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} = \sum_{i=1}^{4}(a_i + jb_i)\,\overline{T}_i\,; \qquad (50)$$

but (3) only presents two complex (four real) linear equations. As posed, the problem is underdetermined and cannot be solved. We may use an optimization technique to infer the true admittance, but since there may be infinite solutions, we can leverage regularization, inequality, and equality constraints:

$$\begin{aligned} \min_{a;b}\ \ & \left\| \tilde{I}\ \ Y\,\tilde{V} \right\|_2^2 + f(a;b;\ ) \\ \text{s.t.}\ \ & Y_a = \sum_{i=1:::4}(a_i + jb_i)\overline{T}_i \\ & 0\ \ g(a;b) \\ & 0 = h(a;b): \end{aligned} \qquad (51)$$

In this section, function h will be used to qualitatively constrain the basis matrices, g will be used to enforce DEF passivity, and f will be used as a regularizing prior (with regularization parameter) which gathers data from other frequency bins outside of the forcing frequency $_d$. After introducing inference techniques to solve load and generator problems, this section shows how the solutions may be used in combination with the perturbative network model to define an improved source-location technique for when the DEF fails.

A. Load Modeling Assumptions

While modeling loads explicitly is challenging, in the context of FOs, we are interested in load response to small, low frequency perturbations. Assuming power is an instantaneous function of voltage magnitude and frequency, we have

$$P(t) = P_0\,(V{=}V_0)^{\ p}\,(!{=}!_0)^{\ p} \qquad (52a)$$

$$Q(t) = Q_0\,(V{=}V_0)^{\ q}\,(!{=}!_0)^{\ q}\,; \qquad (52b)$$

where $! = !_0 + \_$. Since we are interested in the linearized responses of (52a) and (52b), we evaluate their partial derivatives at equilibrium (V_0, $!_0$) to yield the perturbation matrix. Assuming sinusoidal perturbations, phasor notation yields

$$\underbrace{\begin{bmatrix} \tilde{P} \\ \tilde{Q} \end{bmatrix}}_{s} = \underbrace{\begin{bmatrix} {}_pP_0 & {}_pP_0 \\ {}_qQ_0 & {}_qQ_0 \end{bmatrix}}_{Y_a} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & j \end{bmatrix}}_{I_j} \underbrace{\begin{bmatrix} \tilde{V} \\ \tilde{} \end{bmatrix}}_{\tilde{V}_p} \qquad (53)$$

where $\tilde{} = j\,\tilde{}$. Since Y_a is the sum of four real basis matrices, (53) can be solved[5] exactly via the simple inference problem

$$\begin{aligned} \min_{a}\ \ & \left\| S\ \ Y_a I_j \tilde{V}_p \right\|_2^2 \\ \text{s.t.}\ \ & Y_a = \sum_{i=1:::4} a_i\overline{T}_i\,: \end{aligned} \qquad (54)$$

We may use Y_a to compute the admittance which relates voltage and current perturbations. Employing matrix T_1 from [10, eq. (10)], we transform from polar to rectangular in (53):

$$S = Y_a I_j\ \ T_1^{\ 1}T_1\ \tilde{V}_p \qquad (55a)$$

$$= Y_a I_j\,T_1^{\ 1}\tilde{V}\,: \qquad (55b)$$

[3] Since the DWE is the admittance of the network "seen" by the source, the negative DWE is the admittance seen by the network behind the source bus.

[4] The transformed FRF will have one positive and one negative eigenvalue.

[5] Load inference may not be possible when stochastic load variation or measurement noise overpowers the load's dynamic response to the FO.

We consider perturbations of $P=V_rI_r+I_iV_i$ and $Q=V_iI_r-V_rI_i$. Treating $\bar{V}_{i/r}$ and $\bar{I}_{i/r}$ as steady state values, we have

$$\begin{bmatrix}\tilde{P}\\ \tilde{Q}\end{bmatrix}=\underbrace{\begin{bmatrix}\bar{I}_r & \bar{I}_i\\ -\bar{I}_i & \bar{I}_r\end{bmatrix}}_{A_I}\underbrace{\begin{bmatrix}\tilde{V}_r\\ \tilde{V}_i\end{bmatrix}}_{\tilde{\mathbf{V}}}+\underbrace{\begin{bmatrix}\bar{V}_r & \bar{V}_i\\ \bar{V}_i & -\bar{V}_r\end{bmatrix}}_{A_V}\underbrace{\begin{bmatrix}\tilde{I}_r\\ \tilde{I}_i\end{bmatrix}}_{\tilde{\mathbf{I}}}. \tag{56}$$

By equating (55b) and (56), $\tilde{\mathbf{I}}$ and $\tilde{\mathbf{V}}$ are directly related by

$$\tilde{\mathbf{I}}=A_V^{-1}\left(Y_a I_j \mathrm{T}_1^{-1}-A_I\right)\tilde{\mathbf{V}}. \tag{57}$$

B. Generator Modeling Assumptions

Generators are highly complex network elements with multiple control loops and nonlinearities. In response to small, low frequency oscillations, however, the fast electromagnetic transients and the slow droop control effects may often be neglected. We are primarily concerned with a machine's electromechanical response along with any AVR influence. The corresponding generator inference problem which we seek to solve may be biased by any sort of prior model, such as in [14], but eliminating an explicit dependence on an analytically constructed prior model is advantageous for a variety of reasons. There are structural priors which may still be incorporated, though. The FRF of the classical generator, for example, contains no complex variation of basis matrix $\overline{T}_1$, as stated explicitly by (38c). When this model is upgraded to the third order synchronous generator model with first order AVR effects, standard parameter values from [15] may be applied to build its FRF Y_g via (50). We find[6] that

$$\left\|(\alpha_1+j\beta_1)\overline{T}_1\right\|/\|Y_g\|\leq 1\%. \tag{58}$$

Therefore, the effects from basis matrix $\overline{T}_1$ may be safely neglected for a third order machine with an AVR. As a secondary structural constraint, we leverage an important assumption of the DEF method: in response to low frequency oscillations, generators are incrementally passive devices [16]. In the inference problem, we therefore enforce passivity. We also define the set $\Omega_\epsilon=\{\Omega_d-\epsilon,\ldots,\Omega_d,\ldots,\Omega_d+\epsilon\}$, where Ω_d is the forcing frequnecy of interest, and ϵ is the narrow bandwidth around Ω_d where we believe Y_g will have sufficiently little variation (such as $\epsilon=0.05$ Hz). Assuming the system loads are driven by filtered white noise (such as Ornstein Uhlenbeck), the voltage $\tilde{\mathbf{V}}(\Omega)$ and current $\tilde{\mathbf{I}}(\Omega)$ data outside of the forcing frequnecy should present sufficiently rich data for regularizing our generator inference problem:

$$\begin{aligned}\min_{\mathbf{a},\,\mathbf{b}}\;&\left\|\tilde{\mathbf{I}}(\Omega_d)-Y_g\tilde{\mathbf{V}}(\Omega_d)\right\|_2^2+\kappa\sum_{\Omega\in\Omega_\epsilon\setminus\Omega_d}\left\|\tilde{\mathbf{I}}(\Omega)-Y_g\tilde{\mathbf{V}}(\Omega)\right\|_2^2\\ \text{s.t. }\;&Y_g=\sum_{i=2\ldots4}(a_i+jb_i)\,\overline{T}_i\\ &0\leq\lambda\left((\mathbf{M}Y_g\Gamma)+(\mathbf{M}Y_g\Gamma)^{\dagger}\right),\;\forall\lambda\in\{\lambda_1,\lambda_2\}.\end{aligned} \tag{59}$$

While this regularizer is useful for inferring generator admittances, it may not be effective with loads given their stochastic fluctuations. We note that (59) can be solved at a FO source bus and still yield a passive FRF estimate.

[6]The results of (58), along with the passivity assumption of (59), are found to be valid when the effects of the AVR are minor, i.e., when the frequency of the FO is large enough to elicit a small response from the AVR. For very low frequnecy oscillations, passivity is likely to be lost.

C. Comparing Kron Predictions with Inference Results

After solving the generator and load inference problems of the previous two subsections, the perturbative system model in (14) will be explicitly characterized. The location of the FO source, though, will still be unknown since (59) will have been solved at all generator buses: source and non-source. To test which solution to (59) does not correspond to a physically meaningful FRF, the key idea to leverage is this: we assume the inference results are sufficiently correct (physically meaningful) at n_g-1 of the generators, so a physically meaningful DWE can be constructed only at the source. The source DWE will be capable of predicting its own current injection. For robust implementation, we construct a network model similar to (17b) but with two source currents:

$$\begin{bmatrix}-\mathcal{I}'_i\\ -\mathcal{I}'_j\\ \mathbf{0}\end{bmatrix}=\begin{bmatrix}Y_1 & Y_2 & Y_3\\ Y_4 & Y_5 & Y_6\\ Y_7 & Y_8 & Y_9\end{bmatrix}\begin{bmatrix}\mathbf{V}_i\\ \mathbf{V}_j\\ \mathbf{V}_n\end{bmatrix}, \tag{60}$$

where $-\mathcal{I}'_i$, $-\mathcal{I}'_j$ are the measured[7] current injections at generators i,j (meaning their dynamics are not included in the admittance matrix), the admittance matrix is constructed using inference results, and $\mathbf{V}_n$ is the voltage perturbation vector in the remaining network. Since $\mathbf{V}_n=-Y_9^{-1}(Y_7\mathbf{V}_i+Y_8\mathbf{V}_j)$, we may Kron reduce the remaining system:

$$-\mathcal{I}'_i=\left(m_1-m_2m_4^{-1}m_3\right)\mathbf{V}_i-m_2m_4^{-1}\mathcal{I}'_j \tag{61}$$

where $m_1=Y_1-Y_3Y_9^{-1}Y_7$, $m_2=Y_2-Y_3Y_9^{-1}Y_8$, $m_3=Y_4-Y_6Y_9^{-1}Y_7$, and $Y_5-Y_6Y_9^{-1}Y_8$. Thus, we may compare the current prediction of (61) to the actual measured current at this generator. Of particular note is that this current prediction does not depend on the inferred dynamics of generator j, only its current measurement. Since there is one inference result which is not to be trusted, we may sequentially predict the currents of all generators $i\in\{1...n_g\}$ while sequentially removing the dynamics of the other $j\in\{1...n_g\}\setminus i$ generators. When the source generator is removed, prediction error should decrease. Source generator i may be thus identified via

$$\underset{i}{\operatorname{argmin}}\sum_j\left\|\tilde{\mathbf{I}}_i-\left(m_1-m_2m_4^{-1}m_3\right)\mathbf{V}_i+m_2m_4^{-1}\mathcal{I}'_j\right\|_2^2. \tag{62}$$

V. Test Results

For the sake of illustrative expediency, we present a single illustrative test case to portray the validity of the presented framework. Fig. 2 shows a diagram of the IEEE 39-bus New England system. Generators are third order synchronous machines [15] with first order AVRs[8], while the loads are modeled by (52a)-(52b) with Onstein Uhlenbeck noise [14]. For simplicity, measurement noise is not considered, the transmission network model is assumed known, and PMU network observably is full. System excitation and simulation is performed in the frequency domain, and all simulation code is posted online[9]. After computing system response to a 2Hz

[7]Since networks aren't typically fully PMU observable and are often connected to other unobservable networks, vector $-\mathcal{I}'_j$ can also be filled with current injections on the edge of observability, as necessary.

[8]The gains and time constants of these AVRs were chosen to approximate the full regulator + exciter system model from [15].

[9]https://github.com/SamChevalier/Passivity-Enforcement-FOs